

Resilient Opportunistic On-line Global Optimization

Amor Menezes and Pierre Kabamba

Department of Aerospace Engineering

University of Michigan

Ann Arbor, Michigan 48109-2140

amenezes@umich.edu, kabamba@umich.edu

October 29, 2009

Abstract

Traditional off-line global optimization is non-resilient and non-opportunistic. That is, traditional global optimization is unresponsive to small perturbations of the objective function that require a small or large change in the optimizer. On-line optimization methods that are more resilient and opportunistic than their off-line counterparts typically consist of the computationally expensive sequential repetition of off-line techniques. A novel approach to on-line global optimization is to utilize the theory of evolutionary generation systems to develop a technique that is resilient, opportunistic, and inexpensive. The theory of evolutionary generation systems utilizes the probabilistic sequential selection of a candidate optimizer from two possible candidates, basing the selection on the ratio of the fitness values of the candidates and a parameter called the level of selectivity. Using time-homogeneous, irreducible, ergodic Markov chains to model a sequence of local, and hence inexpensive, decisions, this paper proves that such decisions result in the resilient and opportunistic determination of a candidate optimizer for a given objective function. In the limit as the level of selectivity tends to infinity, the theory guarantees that the candidate optimizer is a global optimizer. The optimization of flapping wing gaits illustrates the theory.

Contents

1	Introduction	1
2	Theory of Evolutionary Generation Systems	4
3	Comparative Literature Study	10
3.1	Reinforcement Learning	10
3.2	Simulated Annealing	12
3.3	Genetic Algorithms	12
3.4	Evolutionary Strategies	16
4	Markov Chains That Behave Rationally	17
4.1	Markov Chain Rational Behavior	17
4.2	Entropy of Markov Chains That Behave Rationally	20
4.3	Resiliency and Opportunism of Markov Chains That Behave Rationally	21
5	Selective Evolutionary Generation Systems as Markov Chains That Behave Rationally	24
5.1	Analysis of Selective Evolutionary Generation Systems	24
5.2	Dynamic Properties of Selective Evolutionary Generation Systems	28
6	Illustrative Example of a Selective Evolutionary Generation System	32
6.1	Non-symmetric Matrix of Generation Probabilities	32
6.2	Symmetric Matrix of Generation Probabilities	34
7	Evolution of Flapping Wing Gaits	37
7.1	Surrogate Model	38
7.2	Surrogate Model Results	40
7.3	Theodorsen-Garrick Model	42
7.4	Theodorsen-Garrick Model Results	46
8	Conclusions and Future Work	48
A	Proofs	50

List of Figures

1.1	Off-line optimization strategies yield results that are non-resilient and non-opportunistic.	2
2.1	The directed graph of $G(x, (r_\mu))$	5
6.1	The directed graph of the example in Chapter 6.1.	32
6.2	The directed graph of the example in Chapter 6.2.	34
7.1	Target (dashed black) and actual (solid blue) lift coefficients per generation.	41
7.2	Fitness value per generation.	41
7.3	Flapping wing kinematic parameters per generation.	42
7.4	Snapshots of the forward and back half-strokes of the flapping wing sampled at the 1st, 900th, 1025th, and 2000th generations.	42
7.5	Target trajectory (dashed black), initial trajectory (solid blue) and the 200th evolved (dashed-dotted red) trajectory.	47

Chapter 1

Introduction

This paper is devoted to the problem of *on-line optimization* [1–7], which seeks an optimizer $x^* \in D$ for a real-valued objective function

$$F : D \rightarrow \mathbb{R} : x \rightarrow F(x), \quad (1.1)$$

under the assumption that, during implementation, the user has many motives and opportunities to re-compute a candidate optimizer. This problem is similar to its more traditional off-line counterpart [8–12] in that they both seek an optimizer. However, in *off-line optimization* [13], the candidate optimizer is computed once, and its computation is not revisited during implementation. For this reason, off-line optimization is non-responsive to perturbations of the objective function in two ways: non-resiliency and non-opportunism, which are illustrated in Figs. 1.1 (a) and (b). Specifically, small changes in the objective function may require small changes in the optimizer (which happens when the optimizer depends continuously on the perturbation), or large changes in the optimizer (which happens when the optimizer depends discontinuously on the perturbation), respectively. Hence, the motivation for on-line optimization is that in practice, the objective function on which the candidate optimizer is implemented may be different from that for which the candidate was designed.

The sequential repetition of off-line techniques results in on-line optimization methods that

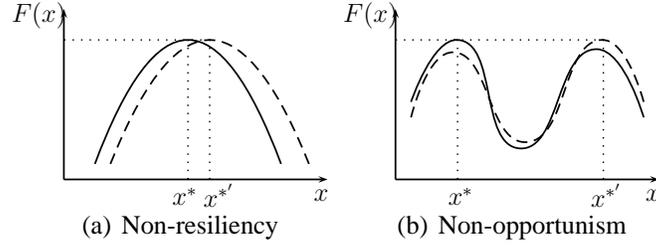


Figure 1.1: Off-line optimization strategies yield results that are non-resilient and non-opportunistic.

are more resilient and opportunistic than their off-line counterparts. However, such sequential repetitions can be computationally expensive, a fact that may be shown by either an amortized analysis [14] or a competitive analysis [1, 2]. Therefore, the goal of this paper is to find an on-line global optimization strategy that is resilient, opportunistic, and inexpensive.

More specifically, this paper shows that *rational behavior* [15] is a sufficient condition for resiliency and opportunism. The work then proposes a resilient and opportunistic, on-line, global optimization scheme based on the novel concept of selective generation, which utilizes the ratio of the fitness values of two candidate optimizers and a parameter called the level of selectivity. In the limit as the level of selectivity tends to infinity, the scheme guarantees that the selected candidate optimizer is a global optimizer. Although rational behavior suggests optimization decisions that are based on global knowledge, this paper proves that rationality may be achieved through a sequence of decisions using only local knowledge of the objective function. Thus, the proposed scheme is also computationally inexpensive at each step.

The original contributions of this work include the following.

- A novel mathematical definition of selection, the *Select* function, for use in probabilistic optimization.
- A demonstration that the canonical genetic algorithm with fitness proportional selection and the (1+1) evolutionary strategy are particular cases of a scheme utilizing the *Select* function.
- A proof that selective generation is a sufficient condition for rational behavior.
- A proof that rational behavior is a sufficient condition for resiliency and opportunism.

- An analysis of the effect that the level of selectivity has on resiliency and opportunism.
- The identification of system-theoretic properties of a selective generation scheme, including equilibria and their stability and optimality properties.
- The evolution of flapping wing parameters for the purpose of trajectory-tracking by a flapping wing vehicle.

The remainder of the paper is as follows. Chapter 2 presents the fundamentals of a theory of evolutionary generation systems that utilizes a novel scheme for fitness-based selection. Chapter 3 documents how evolutionary generation systems are different from other evolutionary computation strategies in the literature. Chapter 4 proves that a sufficient condition for resiliency and opportunism is rational behavior. Chapter 5 furthers the theoretical results by demonstrating that resiliency and opportunism may be achieved inexpensively at each step. Chapter 6 illustrates the theory by means of an analytically treated example. Chapter 7 applies the theory to the optimization of flapping wing gaits. Chapter 8 presents conclusions.

Chapter 2

Theory of Evolutionary Generation Systems

The theory in this chapter is based on concepts from Generation Systems Theory (GST) [16]. GST formalizes the self-reproduction of *cells*, a term describing any entity that is capable of producing an offspring regardless of its physical nature. A robot, a bacterium, or even a piece of software code is considered to be a cell in this theory if they can each produce another robot, bacterium or some lines of code respectively. These cells utilize resources to self-reproduce. A selected resource is manipulated by the parent cell via an embedded generation action to produce an outcome.

We now extend these ideas to develop a theory of evolutionary generation systems. For optimization, a cell is any element of the domain of the objective function (1.1) and a resource is any input that facilitates a transition between cells. Furthermore, it is possible that resources are chosen probabilistically. Consistent with these notions, we make the following definition.

Definition 1. An *evolutionary generation system* is a quadruple $\mathcal{E} = (X, R, P, G)$, where

- X is a set of n cells, $X = \{x_1, x_2, \dots, x_n\}$;
- R is a set of m resources, $R = \{r_1, r_2, \dots, r_m\}$, that can be utilized for cell reproduction;
- $P : R \rightarrow (0, 1]$ is a *probability mass function on R* , given by $P(r_i) = \Pr[\mathcal{R} = r_i] = p_i$,
 $\sum_{k=1}^m p_k = 1$; and

- $G : X \times R \rightarrow X$ is a *generation function* that maps a parent cell and a resource into a descendant cell outcome.

Use of the adjective *evolutionary* here is consistent with biology [17], where evolution is defined as the genetic changes in a biological population that occur every generation due to genetic changes from parent to descendant.

Example 1. A random walk over \mathbb{Z}^ν is an example of an evolutionary generation system. Take $X = \mathbb{Z}^\nu$, $R = \{\pm \mathbf{e}_i, 1 \leq i \leq \nu\}$ (where \mathbf{e}_i are the standard basis vectors for \mathbb{Z}^ν), let P be the uniform probability distribution over R , and define $G : X \times R \rightarrow X : (x, r) \mapsto y = x + r$. The sequence of cells over the generations of this evolutionary generation system becomes a random walk.

Let $(r_\mu) = (r_1, r_2, \dots, r_\mu)$ be a sequence of μ resources from R . We define the notation

$$G(x, (r_\mu)) := G(\dots G(G(x, r_1), r_2) \dots, r_\mu) \quad (2.1)$$

to denote the cell produced by x using sequence (r_μ) . This is illustrated in Fig. 2.1 as a directed graph.

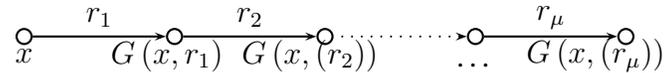


Figure 2.1: The directed graph of $G(x, (r_\mu))$.

Definition 2. The set of cells, X , of the evolutionary generation system $\mathcal{E} = (X, R, P, G)$ is *reachable* through G and R if, for all pairs $x_1, x_2 \in X$, there exists $k \in \mathbb{N}$ and a sequence $(r_k) \in R$ such that $x_2 = G(x_1, (r_k))$.

Note that reachability of the cells of an evolutionary generation system is identical to that of reachability of the vertices of a directed graph in Graph Theory [18].

In Definition 1, the restriction that the offspring of a cell be itself a cell implies that the set of cells is *closed* [19], since there is no feasible transition to any element outside X . If the set of cells

is also reachable, then X is said to be *irreducible* [19]. The previous example of a random walk over \mathbb{Z}^{ν} is an example of an irreducible evolutionary generation system.

We associate each cell with a non-zero, positive performance index that is a measure of the fitness of the cell, $F : X \rightarrow \mathbb{R}^+$. The notion of fitness facilitates the following novel mathematical definition of selection.

Definition 3. Given a cell set, X , and a fitness function $F : X \rightarrow \mathbb{R}^+$, let $Select : X \times X \times \mathbb{N} \rightarrow X$ be a random function such that if $x_1, x_2 \in X$ are any two cells, and $N \in \mathbb{N}$ is the *level of selectivity*, then

$$Select(x_1, x_2, N) = \begin{cases} x_1 & \text{with probability } \frac{1}{1 + \left(\frac{F(x_2)}{F(x_1)}\right)^N}, \\ x_2 & \text{with probability } \frac{1}{1 + \left(\frac{F(x_1)}{F(x_2)}\right)^N}. \end{cases} \quad (2.2)$$

We can now define a selective evolutionary generation system (SEGS).

Definition 4. A *selective evolutionary generation system* is a quintuple $\Gamma = (X, R, P, G, F)$, where

- (X, R, P, G) is an evolutionary generation system;
- $F : X \rightarrow \mathbb{R}^+$ is a function that evaluates cell fitness;
- the set of cells, X , is reachable through G and R ; and
- the dynamics of the system are given by

$$\mathcal{X}(t+1) = Select(\mathcal{X}(t), G(\mathcal{X}(t), \mathcal{R}(t)), N). \quad (2.3)$$

In (2.3), $\mathcal{X}(t)$ denotes the realization of a random cell variable at time t , $\mathcal{R}(t)$ denotes the realization of a random resource variable at time t , $G(\mathcal{X}(t), \mathcal{R}(t))$ denotes the offspring of the realized random cell utilizing the realized random resource at time t , and $\mathcal{X}(0)$ has a known probability mass function.

Also in (2.3), the probability of a cell realization at some future time given the present cell realization is conditionally independent of the past time history of cell realizations. Thus, the dynamics of a SEGS form a discrete-time homogeneous Markov chain [20]. This property is useful for the SEGS analysis conducted in Chapter 5.

The two central tenets of Darwin's theory of evolution [17] are embodied in Definition 4.

1) *Undirected variation* via the generation function. Permissible undirected variations include

- *mutations* of all or part of a cell,
- *recombination* of the constituent elements of a cell with the constituent elements of another (resource) cell,
- *inheritance* of all or part of a cell when the generation function maps all or part of a cell to itself, and
- *drift* of the constituent elements of a cell, as certain elements drift or become fixed due to the nature of the probability mass function over the resource set.

That is, we impose no restrictions on the nature of the undirected variation process, thereby capturing all biological and computational mechanisms for creating diverse offspring. As we shall see, even *flow*, the sudden addition or removal of cells, is captured by our theory, since this process may be modeled by unexpected perturbations of the fitness function.

2) *Natural selection* via the *Select* function.

The *Select* function has a number of interesting properties, including:

- For all N ,

$$\frac{\Pr[\text{Select}(x_1, x_2, N) = x_1]}{\Pr[\text{Select}(x_1, x_2, N) = x_2]} = \left(\frac{F(x_1)}{F(x_2)} \right)^N. \quad (2.4)$$

That is, the ratio of the probabilities of selecting any two cells is equal to the ratio of their respective fitnesses raised to the power N .

- For $N = 0$, the values of $F(x_1)$ and $F(x_2)$ are irrelevant. That is,

$$\Pr[\text{Select}(x_1, x_2, 0) = x_1] = 1/2, \text{ and} \quad (2.5)$$

$$\Pr[\text{Select}(x_1, x_2, 0) = x_2] = 1/2. \quad (2.6)$$

- When $N = \infty$, if $F(x_1) > F(x_2)$ then

$$\Pr[\text{Select}(x_1, x_2, \infty) = x_1] = 1. \quad (2.7)$$

On the other hand, if $F(x_1) < F(x_2)$ then

$$\Pr[\text{Select}(x_1, x_2, \infty) = x_2] = 1. \quad (2.8)$$

- If $F(x_1) = F(x_2)$ then, for all N ,

$$\Pr[\text{Select}(x_1, x_2, N) = x_1] = 1/2, \text{ and} \quad (2.9)$$

$$\Pr[\text{Select}(x_1, x_2, N) = x_2] = 1/2. \quad (2.10)$$

The level of selectivity, N , has a biological interpretation as well. Suppose that the fitness of a cell is measured by the total number of descendants produced over k generations, $k \geq 1$. This prolificity is typically called *future reproductive value* or *fecundity* [17]. When a colony is initiated by two self-reproducing progenitors x_1 and x_2 , the ratio of the descendant population fractions after k generations equals the ratio of the respective future reproductive values,

$$\left(\frac{F(x_1)}{F(x_2)} \right). \quad (2.11)$$

After k generations, the ratio of the probability of choosing, by random sampling, a descendant of x_1 to the probability of choosing a descendant of x_2 is equal to the ratio of the descendant

population fractions, (2.11). Correspondingly, the ratio of the probability of selecting x_1 at the initial time to the probability of selecting x_2 at the initial time, (2.4), is identical to the ratio of the respective prolificities, (2.11), with $N = 1$.

Now consider the following sequence of operations.

1. Initiate a colony with two self-reproducing progenitors x_1 and x_2 , and let descendants be produced for k generations.
2. Extract a sample from the resulting population. Use the sample to initiate a second colony, and let descendants be produced for k generations.
3. Iterate the sample and colony initiation procedure until an N^{th} colony is produced.

Then, the ratio of the probability of selecting a descendant of x_1 to the probability of selecting a descendant of x_2 using this multi-step process becomes

$$\left(\frac{F(x_1)}{F(x_2)}\right) \left(\frac{F(x_1)}{F(x_2)}\right) \cdots \left(\frac{F(x_1)}{F(x_2)}\right) = \left(\frac{F(x_1)}{F(x_2)}\right)^N, \quad (2.12)$$

and it is now clear that N represents the number of selections that are made, assuming a k -generation fecundity interpretation of fitness.

A recent, well-publicized, biological experiment that fits this multi-selection model is [21]. Two polyethylene degrading strains of bacteria were isolated in this study as a result of the repeated selections of the progeny of soil bacteria that were forced to feed on a polyethylene enriched medium.

Chapter 3

Comparative Literature Study

A SEGS as described by Definition 4 can be utilized as an evolutionary optimization algorithm (Chapter 5) to take advantage of its guaranteed properties. There are works in the optimization literature that appear to be similar: reinforcement learning [22], simulated annealing [23,24], genetic algorithms [25–27], and evolutionary strategies [28–31]. Comparisons between these optimization methodologies and a SEGS approach can be made, and this chapter is devoted to providing such comparisons to outline the distinctions between approaches. For each of the optimization methodologies, we quantify the ratio of the probability of selecting a candidate optimizer of (1.1) to the probability of selecting the optimizer’s offspring. By comparing this resultant ratio to (2.4), we demonstrate the originality of our theory of evolutionary generation systems.

3.1 Reinforcement Learning

In reinforcement learning (RL) [22], a decision-making agent takes actions in an environment and receives a corresponding reward. The traditional RL problem is to determine the best policy or sequence of actions that maximizes the total reward. There are two major differences between our work and RL. First, evolutionary generation systems theory does not allow changes in the tactics of individual cells, since there are fixed, probabilistic rules for generation outcomes in place. That is, the generation action taken by a particular cell is always the same but the outcome

varies probabilistically due to selection. Second, RL seeks the long-term maximization of reward of a policy of cell-action pairs, while a SEGS focuses on probabilistically increasing the short-term reward from one cell-action pair. The different goals have consequences for responsiveness: an RL approach may not adapt the optimal policy if individual cell-action pair rewards are perturbed.

To facilitate a comparison between RL and a SEGS, consider the following deterministic reinforcement learning problem. Let x_1 and x_2 be the labels of two terminal cells, and let the current cell, also labeled x_1 , be capable of a one-step transition to either of the two terminal cells. Hence, there are two possible policies: 1) a transition from x_1 to x_1 , and 2) a transition from x_1 to x_2 . Let the reward of cell x_i be $F(x_i)$. Using value iteration, the cost-to-go of the current cell with policy 1) is

$$V_1 = F(x_1) + F(x_1) = 2F(x_1), \quad (3.1)$$

and the cost-to-go of the current cell with policy 2) is

$$V_2 = F(x_1) + F(x_2). \quad (3.2)$$

Since RL chooses the policy with maximum reward, the ratio of the probability of selecting the terminal cell x_1 to the probability of selecting the terminal cell x_2 is

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} = \frac{\text{ind}(2F(x_1) > F(x_1) + F(x_2))}{\text{ind}(F(x_1) + F(x_2) > 2F(x_1))}, \quad (3.3)$$

where ind denotes the indicator function, satisfying

$$\text{ind}(True) = 1, \quad (3.4)$$

$$\text{ind}(False) = 0. \quad (3.5)$$

This ratio is different from (2.4).

3.2 Simulated Annealing

The simulated annealing algorithm [24] randomly samples the search space at x_i , evaluates $F(x_i)$, and accepts new candidate optimizers x_i according to the Metropolis criterion. This criterion specifies that cells with better fitness are always accepted, while less fit cells are accepted with a probability that depends on the relative fitness with respect to the current cell x_1 , and a “temperature” parameter T . The equation for the probability of selecting the less fit cell, x_2 , is

$$\Pr[x_2 \text{ is selected}] = p_{SA} = \exp\left(\frac{F(x_2) - F(x_1)}{T}\right), \quad (3.6)$$

whenever $F(x_1) > F(x_2)$. Therefore, decreasing the temperature or increasing the relative fitness decreases the acceptance probability of less fit states.

The ratio of the probability of selecting the current candidate optimizer x_1 to the probability of selecting another candidate optimizer x_2 is

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} = \frac{1 - (\text{ind}(F(x_2) > F(x_1)) + \text{ind}(F(x_1) > F(x_2))) p_{SA}}{\text{ind}(F(x_2) > F(x_1)) + \text{ind}(F(x_1) > F(x_2)) p_{SA}}. \quad (3.7)$$

This ratio is different from (2.4).

3.3 Genetic Algorithms

The canonical genetic algorithm [25] models each cell of the search space, x_i , as a binary string of length l to which a fitness value $F(x_i)$ is associated. The algorithm outline [32] follows:

- 1: choose an initial population
- 2: determine the fitness of each individual
- 3: perform selection
- 4: **repeat**

- 5: perform crossover
- 6: perform mutation
- 7: determine the fitness of each individual
- 8: perform selection
- 9: **until** some stopping criterion applies

We are interested in the probability that a cell, x_1 , of the population at Line 4 is chosen to be a member of the population for the next generation (i.e., after one iteration of the repeat loop) without experiencing crossover or mutation. We then compare this probability to the probability that an offspring of x_1 is a member of the population at the next generation. Let the probability of crossover of x_1 with another binary string be $p_c \in (0, 1)$, and let mutation of the j -th bit of x_1 occur independently with probability $p_m \in (0, 1)$.

Of the many kinds of selection processes (e.g., fitness-proportional selection, tournament selection, or truncation selection) that can be applied to the cells of a population, Y , let us first consider fitness-proportional selection. The probability of considering x_1 with this selection process is

$$\Pr[x_1 \text{ is considered}] = \frac{F(x_1)}{\sum_{y \in Y} F(y)}. \quad (3.8)$$

Hence, the probability that an unchanged candidate optimizer, x_1 , is a member of the population for the next generation is

$$\Pr[x_1 \text{ is selected}] = (1 - p_c)(1 - p_m)^l \frac{F(x_1)}{\sum_{y \in Y} F(y)}. \quad (3.9)$$

If x_2 is an offspring of x_1 that undergoes crossover with probability p_c , undergoes mutation of k bits with probability p_m , and is subjected to fitness-proportional selection, it becomes a member of the population for the next generation with probability

$$\Pr[x_2 \text{ is selected}] = p_c p_m^k (1 - p_m)^{l-k} \frac{F(x_2)}{\sum_{y \in Y} F(y)}. \quad (3.10)$$

Thus, the ratio of the probability of selecting x_1 to the probability of selecting x_2 becomes

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} = \frac{(1 - p_c) (1 - p_m)^k F(x_1)}{p_c p_m^k F(x_2)} = K \frac{F(x_1)}{F(x_2)}, \quad K > 0. \quad (3.11)$$

Although the equation above is similar to (2.4), it demonstrates that in the canonical genetic algorithm using fitness-proportional selection, the ratio of selection probabilities is proportional to the fitness ratio. In (3.11), if $K = 1$ we obtain a particular case of (2.4) where $N = 1$.

With tournament selection, the probability of considering x_1 is

$$\Pr[x_1 \text{ is considered}] = \begin{cases} p_s \in (0, 1), & \text{if } \forall y \in Y, F(x_1) > F(y), \\ p_s(1 - p_s), & \text{if } |\{y \in Y \mid F(y) > F(x_1)\}| = 1, \\ p_s(1 - p_s)^2, & \text{if } |\{y \in Y \mid F(y) > F(x_1)\}| = 2, \\ \dots & \dots \\ p_s(1 - p_s)^{|Y|-1}, & \text{if } |\{y \in Y \mid F(y) > F(x_1)\}| = |Y| - 1. \end{cases} \quad (3.12)$$

Using $\text{rank}(x_1)$ to denote $|\{y \in Y \mid F(y) > F(x_1)\}|$, the probability that an unchanged candidate optimizer, x_1 , is a member of the population for the next generation is

$$\Pr[x_1 \text{ is selected}] = (1 - p_c)(1 - p_m)^l p_s(1 - p_s)^{\text{rank}(x_1)}. \quad (3.13)$$

If x_2 is an offspring of x_1 that undergoes crossover with probability p_c , undergoes mutation of k bits with probability p_m , and is subjected to tournament selection, it becomes a member of the population for the next generation with probability

$$\Pr[x_2 \text{ is selected}] = p_c p_m^k (1 - p_m)^{l-k} p_s(1 - p_s)^{\text{rank}(x_2)}. \quad (3.14)$$

Thus, the ratio of the probability of selecting x_1 to the probability of selecting x_2 becomes

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} = \frac{(1 - p_c) (1 - p_m)^k (1 - p_s)^{\text{rank}(x_1)}}{p_c p_m^k (1 - p_s)^{\text{rank}(x_2)}} = K(1 - p_s)^{\text{rank}(x_1) - \text{rank}(x_2)}, \quad K > 0. \quad (3.15)$$

This result is different from (2.4).

With truncation selection, the probability of considering x_1 is

$$\Pr[x_1 \text{ is considered}] = \text{ind} \left(\text{rank}(x_1) < \frac{|Y|}{2} \right). \quad (3.16)$$

Hence, the probability that an unchanged candidate optimizer, x_1 , is a member of the population for the next generation is

$$\Pr[x_1 \text{ is selected}] = (1 - p_c)(1 - p_m)^l \text{ind} \left(\text{rank}(x_1) < \frac{|Y|}{2} \right). \quad (3.17)$$

If x_2 is an offspring of x_1 that undergoes crossover with probability p_c , undergoes mutation of k bits with probability p_m , and is subjected to truncation selection, it becomes a member of the population for the next generation with probability

$$\Pr[x_2 \text{ is selected}] = p_c p_m^k (1 - p_m)^{l-k} \text{ind} \left(\text{rank}(x_2) < \frac{|Y|}{2} \right). \quad (3.18)$$

Thus, the ratio of the probability of selecting x_1 to the probability of selecting x_2 becomes

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} =$$

$$\frac{(1 - p_c) (1 - p_m)^k \text{ind} \left(\text{rank} (x_1) < \frac{|Y|}{2} \right)}{p_c p_m^k \text{ind} \left(\text{rank} (x_2) < \frac{|Y|}{2} \right)}, \quad (3.19)$$

$$= K \frac{\text{ind} \left(\text{rank} (x_1) < \frac{|Y|}{2} \right)}{\text{ind} \left(\text{rank} (x_2) < \frac{|Y|}{2} \right)}, \quad K > 0. \quad (3.20)$$

This ratio is different from (2.4).

3.4 Evolutionary Strategies

Like genetic algorithms, the general evolutionary strategy [30] operates on a population of cells of the search space of (1.1). Typically, a parent population of size μ creates an offspring population of size λ using crossover and mutation processes, with ρ parent cells required to produce one offspring. Crossover does not occur if $\rho = 1$. Selection of the population for the next generation of the algorithm occurs by picking the μ best cells from the λ offspring (known as the $(\mu/\rho, \lambda)$ -ES strategy), or by picking the μ best cells from the total population of $\mu + \lambda$ cells (known as the $(\mu/\rho + \lambda)$ -ES strategy).

The $(1 + 1)$ -ES strategy [33] is most similar to evolutionary generation systems theory. Here, one candidate optimizer, x_1 , produces one mutated offspring candidate optimizer, x_2 , and the ratio of the probability of selecting x_1 to the probability of selecting x_2 is simply

$$\frac{\text{Pr}[x_1 \text{ is selected}]}{\text{Pr}[x_2 \text{ is selected}]} = \frac{\text{ind} (F (x_1) > F (x_2))}{\text{ind} (F (x_2) > F (x_1))}. \quad (3.21)$$

This ratio equals (2.4) when the parameter N in (2.4) is ∞ .

Chapter 4

Markov Chains That Behave Rationally

In this chapter, we develop a Theory of Rational Behavior [15] for time-homogeneous, irreducible, ergodic Markov chains. We then discuss the entropy, resiliency and opportunism of Markov chains that satisfy the axioms of this theory.

4.1 Markov Chain Rational Behavior

Let (X, \mathbf{P}) be a time-homogeneous, irreducible, ergodic Markov chain, where $X = \{x_1, x_2, \dots, x_n\}$ is the set of states of a Markov process, $\mathbf{P} \in \mathbb{R}^{n \times n}$ is the matrix of transition probabilities for these states, and $n < \infty$ is the number of states. Assume that the initial probability distribution over the states is known, i.e., we are given an n -vector $\mathbf{p}(0)$ having elements $p_i(0) = \Pr[\mathcal{X}(0) = x_i]$ for all $x_i \in X$, where $\mathcal{X}(0)$ denotes the state realization at time 0, and we have $\sum_{i=1}^n p_i(0) = 1$. Since we have assumed that the states in X are ergodic and irreducible, they admit a unique stationary probability distribution [19, 20]. Let $\boldsymbol{\pi} = [\pi_1 \ \pi_2 \ \dots \ \pi_n]$ be the row vector of these stationary probabilities, satisfying the constraints $\pi_i > 0 \ \forall i$, and $\sum_{i=1}^n \pi_i = 1$. Let $F : X \rightarrow \mathbb{R}^+$ be a positive fitness function. Let $N \in \mathbb{N}$ be a natural number. We define rational behavior for this Markov chain as follows.

Definition 5. The time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) is said to *behave*

rationality with respect to fitness F with level N if

$$\frac{\pi_i}{\pi_j} = \left(\frac{F(x_i)}{F(x_j)} \right)^N, \quad 1 \leq i, j \leq n. \quad (4.1)$$

This definition is consistent with [15] because time averages and ensemble averages are equal in an ergodic process. The requirement that $\pi_i > 0 \forall i$ with $\sum_{i=1}^n \pi_i = 1$ corresponds to the ergodic postulate of [15], and the requirement that $N > 0$ corresponds to the selective (i.e., retardation) postulate. Note that we have recast the requisite scalar function of [15] as a reward, instead of a penalty.

Each stationary probability can also be explicitly characterized to ensure Markov chain rational behavior, as is indicated by the following theorem.

Theorem 1. *The time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) behaves rationally with respect to fitness F with level N if and only if*

$$\pi_i = \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N}, \quad 1 \leq i \leq n. \quad (4.2)$$

Proof. See Appendix. □

Here, we have a more general, probabilistic version of the optimization in (1.1). A Markov chain that behaves rationally will select the state of maximum fitness with the highest stationary probability, and, in the limit as N approaches ∞ , this probability is 1. The problem and solution then revert to one of standard optimization.

Remarkably, rational behavior in Markov chains is the result of a subsidiary optimization.

Theorem 2. *The stationary distribution $\boldsymbol{\pi}$ of the time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) that behaves rationally with respect to fitness F with level N solves the optimization problem*

$$\min_{\pi_1, \dots, \pi_n} \Phi(\boldsymbol{\pi}) = - \sum_{i=1}^n F(x_i)^N \ln(\pi_i), \quad (4.3)$$

subject to the constraints

$$\sum_{i=1}^n \pi_i = 1, \quad (4.4)$$

$$\pi_i > 0, \forall i. \quad (4.5)$$

Proof. See Appendix. □

Note that in (4.1), rational behavior is invariant under positive scaling of fitness. Hence, there is no loss of generality in assuming that the fitness function is normalized. Accordingly, let $\varphi = [\varphi_1 \ \varphi_2 \ \dots \ \varphi_n]$ be the distribution of the N^{th} power of fitness, where

$$\varphi_i = \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N}, \quad 1 \leq i \leq n. \quad (4.6)$$

Definition 6. A vector $v \in \mathbb{R}^n$ is a *positive mass function of order n* if it satisfies $v_i > 0 \forall i$, and $\sum_{k=1}^n v_k = 1$. Let \mathbb{D}_n be the set of positive mass functions of order n .

The vector $\varphi \in \mathbb{R}^n$ is a positive mass function. Let

$$U(\boldsymbol{\pi}) = \frac{\Phi(\boldsymbol{\pi})}{\sum_{k=1}^n F(x_k)^N}. \quad (4.7)$$

Then, the optimization problem (4.3) can be normalized as

$$\min_{\pi_1, \dots, \pi_n} U(\boldsymbol{\pi}) = - \sum_{i=1}^n \varphi_i \ln(\pi_i), \quad (4.8)$$

subject to the constraints (4.4) and (4.5). Furthermore, Theorem 2 states that at the optimum, the stationary distribution agrees with the fitness distribution, i.e., $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

4.2 Entropy of Markov Chains That Behave Rationally

Definition 7. *Entropy* [34] is the function

$$H : \mathbb{D}_n \rightarrow \mathbb{R} : \varphi \mapsto H(\varphi) = - \sum_{i=1}^n \varphi_i \ln(\varphi_i). \quad (4.9)$$

Using the notion of entropy, we can interpret (4.8) as follows. First, we recognize the term $-\ln(\pi_i)$ as the information content of state x_i [34]. Hence, the right hand side of (4.8) represents the “fitness-expectation of information.” Moreover, we have the following:

Corollary 1. *The time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) behaves rationally with respect to fitness F with level N if and only if its stationary probability distribution minimizes the fitness-expectation of information. At the optimum, this fitness-expectation of information is the entropy of the fitness distribution, i.e.,*

$$U^* = H(\varphi) = - \sum_{i=1}^n \varphi_i \ln(\varphi_i). \quad (4.10)$$

A basic property of entropy that is alluded to in [35] and which will be utilized in the proof of Theorem 12 follows.

Theorem 3. *Let $\varphi \in \mathbb{D}_n$ be arbitrary. Then,*

$$\min_{\pi \in \mathbb{D}_n} - \sum_{i=1}^n \varphi_i \ln(\pi_i), \quad (4.11)$$

has a minimum value of $H(\varphi)$ that is achieved at $\pi = \varphi$.

Equivalently, $\forall \varphi \in \mathbb{D}_n, \forall \pi \in \mathbb{D}_n,$

$$- \sum_{i=1}^n \varphi_i \ln(\pi_i) \geq - \sum_{i=1}^n \varphi_i \ln(\varphi_i), \quad (4.12)$$

with the equality holding if and only if $\pi = \varphi$.

Equivalently, $\forall \varphi \in \mathbb{D}_n, \boldsymbol{\pi} \in \mathbb{D}_n,$

$$-\sum_{i=1}^n \varphi_i \ln \left(\frac{\pi_i}{\varphi_i} \right) \geq 0, \quad (4.13)$$

with the equality holding if and only if $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Proof. See Appendix. □

For Markov chains that behave rationally, and therefore possess fitness fractions that are distributed over the set of states as in (4.6), the entropy quantifies how egalitarian or elitist the states are. That is, the entropy is highest when all states have equal fitness; conversely, the entropy is lowest when there is only one state with a fitness fraction of unity and all other fitness fractions are zero. Equation (4.9) arises in other well-known fields, and similar interpretations for the distributed quantities and the entropy exist [34–37].

4.3 Resiliency and Opportunism of Markov Chains That Behave Rationally

We can now formally define resiliency and opportunism, first described through Fig. 1.1, as the sensitivity of the stationary distribution to changes in fitness.

Definition 8. For any time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) with a positive fitness function for all the states in X , the *extrinsic resiliency* of state x_i to changes in the fitness of state $x_j, j \neq i$, is defined as

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)}, \quad (4.14)$$

and the *intrinsic resiliency* of state x_i to changes in its own fitness is taken to be

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)}. \quad (4.15)$$

Since the stationary distribution π has the closed form expression (4.2) for the time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) that behaves rationally with respect to fitness F with level N , the extrinsic and intrinsic resiliencies are

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)} = \frac{-N\pi_i\pi_j}{F(x_j)}, \quad \forall j \neq i, \quad (4.16)$$

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)} = \frac{N\pi_i(1 - \pi_i)}{F(x_i)}. \quad (4.17)$$

We say that the Markov chain (X, \mathbf{P}) is *resilient and opportunistic* if $\rho_{ij} \neq 0$ for all i and j .

The level of selectivity has the following asymptotic effect on resiliency and opportunism.

Theorem 4. *For the time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) that behaves rationally with respect to fitness F with level N ,*

$$\rho_{ij} \Big|_{\substack{N=0 \\ j \neq i}} = \rho_{ii} \Big|_{N=0} = 0, \quad (4.18)$$

and

$$\lim_{\substack{N \rightarrow \infty \\ j \neq i}} \rho_{ij} = \lim_{N \rightarrow \infty} \rho_{ii} = 0. \quad (4.19)$$

Proof. See Appendix. □

As a result of Theorem 4, we have quantification that standard optimization ($N = \infty$) is non-resilient. Moreover, recall that if we assume a k -generation fecundity interpretation of fitness as in Chapter 2, then $N = \infty$ also represents an infinite number of selections made over k generations. There is much biological evidence to confirm that prolonged selective breeding yields non-resilient strains [38–42].

Resiliency and opportunism is a direct outcome of Markov chain rational behavior, as stated below.

Theorem 5. *The time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) is resilient and opportunistic if the chain behaves rationally.*

Proof. See Appendix. □

Resiliency and opportunism do not always imply Markov chain rational behavior (see Chapter 6). But we can state the following instead.

Theorem 6. *Ergodicity is a necessary condition for the time-homogeneous, irreducible Markov chain (X, \mathbf{P}) to be resilient and opportunistic.*

Proof. See Appendix. □

Furthermore, there is a fundamental trade-off between extrinsic and intrinsic resiliency that is imposed by the constraint $\sum_{i=1}^n \pi_i = 1$. Taking the partial derivative of this constraint with respect to the fitness of state x_i , we obtain

$$\frac{\partial \pi_i}{\partial F(x_i)} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\partial \pi_j}{\partial F(x_i)} = 0. \quad (4.20)$$

Note that, from (4.16) and (4.17), the extrinsic resiliencies are always negative, whereas the intrinsic resiliencies are positive. Hence, (4.20) implies that any change in fitness that improves a state's intrinsic resiliency is at the expense of the extrinsic resiliency of all other states. Similarly, any change in fitness that improves a state's extrinsic resiliency is at the expense of the intrinsic resiliency of another state, and the extrinsic resiliency of all other states.

Chapter 5

Selective Evolutionary Generation Systems as Markov Chains That Behave Rationally

This chapter applies the Theory of Rational Behavior for time-homogeneous, irreducible, ergodic Markov chains (as developed in Chapter 4) to a SEGS as formulated in Chapter 2. We begin with some preliminaries.

5.1 Analysis of Selective Evolutionary Generation Systems

Definition 9. Let $\Gamma = (X, R, P, G, F)$ be a selective evolutionary generation system. Let $x_i, x_j \in X$ be any two cells, and $r_k \in R$ be a resource. The *descendancy tensor*, δ , has elements

$$\delta_{ijk} = \begin{cases} 1 & \text{if } x_j = G(x_i, r_k), 1 \leq i, j \leq n, 1 \leq k \leq m, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1)$$

Hence, the descendancy tensor indicates whether it is possible to produce cell x_j in one step from cell x_i , using resource r_k . We can use this tensor to create a matrix that represents the conditional probability of generating x_j given that the progenitor is x_i , by utilizing the probability of selecting each available resource and summing over all m resources as follows.

Definition 10. For the SEGS $\Gamma = (X, R, P, G, F)$, the *matrix of generation probabilities*, γ , also called the unselective matrix of transition probabilities, has elements

$$\gamma_{ij} = \Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i], \quad (5.2)$$

$$= \sum_{k=1}^m \delta_{ijk} p_k, \quad 1 \leq i, j \leq n. \quad (5.3)$$

This matrix is a stochastic matrix, as indicated by the following lemma.

Lemma 1. For the SEGS $\Gamma = (X, R, P, G, F)$ with matrix of generation probabilities γ ,

$$\sum_{j=1}^n \gamma_{ij} = 1, \quad 1 \leq i \leq n. \quad (5.4)$$

Proof. See Appendix. □

Recall that a SEGS follows the stochastic Markov process described by (2.3). Therefore, we can find a matrix of transition probabilities to describe the cell-to-cell transitions that occur as a result of the selection dynamics. For the SEGS $\Gamma = (X, R, P, G, F)$, the *matrix of transition probabilities*, \mathbf{P} , has elements

$$P_{ij} = \Pr[\mathcal{X}(t+1) = x_j \mid \mathcal{X}(t) = x_i], \quad (5.5)$$

$$= \Pr[\text{Select}(x_i, x_j, N) = x_j \mid \mathcal{X}(t) = x_i] \times$$

$$\Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i] \quad (5.6)$$

$$= \begin{cases} \frac{1}{1 + \left(\frac{F(x_i)}{F(x_j)}\right)^N} \gamma_{ij}, & \forall j \neq i, \\ \gamma_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \gamma_{ij}, & \text{if } j = i. \end{cases} \quad (5.7)$$

Note that the matrix of transition probabilities in (5.7) is also a stochastic matrix:

Theorem 7. For the SEGS $\Gamma = (X, R, P, G, F)$ with matrix of transition probabilities \mathbf{P} ,

$$\sum_{j=1}^n P_{ij} = 1, \quad 1 \leq i \leq n. \quad (5.8)$$

Proof. See Appendix. □

In addition to irreducibility, if we assume that the selection dynamics of the SEGS is ergodic, then a unique stationary probability distribution over the set of cells exists, and must satisfy the following.

Theorem 8. For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, let $\boldsymbol{\pi} = [\pi_1 \ \pi_2 \ \dots \ \pi_n]$ be the row vector of stationary probabilities, satisfying $\sum_{i=1}^n \pi_i = 1$. Assume that there is a unique index, I , such that $F(x_i)$ is maximized for $i = I$. Then,

$$\lim_{N \rightarrow \infty} \pi_i = 0, \quad 1 \leq i \leq n, \quad i \neq I, \quad (5.9)$$

$$\lim_{N \rightarrow \infty} \pi_I = 1. \quad (5.10)$$

Proof. See Appendix. □

It is easy to extend this theorem and its proof to the case where I is not unique and show that the cells with equal maximal fitness are equiprobable. For both versions of the theorem, there exist stationary probabilities equal to zero in the limit as N approaches ∞ because the stochastic selection process becomes elitist instead of ergodic.

We can also examine the SEGS response to changes in selectivity and cell fitness. First, the probability of increasing fitness with every time step, conditioned upon knowledge of the current cell, is

$$\begin{aligned} & \Pr [F(\mathcal{X}(t+1)) > F(\mathcal{X}(t)) \mid \mathcal{X}(t) = x_i] \\ &= \sum_{\substack{j=1 \\ j \neq i}}^n \text{ind}(F(x_j) > F(x_i)) P_{ij}. \end{aligned} \quad (5.11)$$

This conditional probability increases as N increases. However, the unconditional probability of increasing fitness with every time step,

$$\begin{aligned} & \Pr [F(\mathcal{X}(t+1)) > F(\mathcal{X}(t))] \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \text{ind}(F(x_j) > F(x_i)) P_{ij} \pi_i, \end{aligned} \quad (5.12)$$

approaches zero in the limit as N approaches infinity. That is, the unconditional probability decreases as N increases. This (perhaps counter-intuitive) result is due to the elitist nature of the resultant selection process — the cell with maximal fitness has a stationary probability of 1, and consequently, the probability of improving fitness is correspondingly 0.

Next, the effect of changes in cell fitness on elements of the matrix of transition probabilities, \mathbf{P} , is given by the following four equations:

$$\forall j \neq i, \frac{\partial P_{ij}}{\partial F(x_j)} = \frac{N}{F(x_j)} \left(\frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \right) P_{ij}, \quad (5.13)$$

$$\frac{\partial P_{ii}}{\partial F(x_j)} = \frac{-N}{F(x_j)} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} P_{ij}, \quad (5.14)$$

$$\forall j \neq i, \frac{\partial P_{ij}}{\partial F(x_i)} = \frac{-N}{F(x_i)} \left(\frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \right) P_{ij}, \quad (5.15)$$

$$\frac{\partial P_{ii}}{\partial F(x_i)} = \frac{N}{F(x_i)} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} P_{ij}. \quad (5.16)$$

In the first equation above, we see that an increase in the fitness of cell x_j increases the probability of transitioning to that cell from current cell x_i by an amount that is proportional to the level of selectivity and inversely proportional to the fitness value. The second equation indicates a corresponding decrease in the probability of transitioning back to the current cell under the same altered fitness landscape. Unlike gradient ascent optimization where the transition to another cell would be directly proportional to the fitness value, what we have here is reminiscent of the retardation

property in [15]; the stochastic process “slows down” transitions in more favorable fitness conditions to take advantage of the external environment. Similar effects on the transition probabilities are suggested by the latter two equations for changes in current cell fitness.

5.2 Dynamic Properties of Selective Evolutionary Generation Systems

We can now state some intriguing dynamic properties of selective evolutionary generation systems, under certain technical conditions.

Theorem 9. *For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, assume that the matrix of generation probabilities, γ , is symmetric. Then the Markov chain representing the stochastic dynamics of the ergodic SEGS behaves rationally with fitness F and level N . That is, the row vector $\pi = [\pi_1 \ \pi_2 \ \dots \ \pi_n]$, where π_i satisfies (4.2), is a left eigenvector of \mathbf{P} , the matrix of transition probabilities for Γ , with corresponding eigenvalue 1 (i.e., $\pi\mathbf{P} = \pi$). Hence, π is the vector of stationary probabilities for the SEGS.*

Proof. See Appendix. □

As a result of Theorem 5, the stochastic dynamics of the ergodic SEGS with symmetric matrix of generation probabilities, γ , are resilient and opportunistic. Hence, a SEGS is a computationally inexpensive on-line technique to achieve these characteristics because only local decisions between two candidate optimizers are made at any time. The need to evaluate the fitness of all elements in the domain of the objective function (1.1), or even in a sub-population of candidate optimizers (as in genetic algorithms or evolutionary strategies), is avoided.

The symmetry condition on the matrix of generation probabilities, γ , implies that there exists an equiprobable forward and reverse transition between any pair of cells prior to the selection process. More specifically, symmetry of γ is a requirement that mutations be reversible. This

reversibility requirement is satisfied in biology, and such mutations are called *true back mutations* [43,44].

Theorem 10. *For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, assume that the matrix of generation probabilities, γ , is symmetric. Then the Markov chain representing the stochastic dynamics of the ergodic SEGS is time-reversible, i.e.,*

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad \forall i, j. \quad (5.17)$$

Proof. See Appendix. □

As a consequence, the Markov chain representing the stochastic dynamics of the SEGS and its time reversed form are statistically the same.

Theorem 11. *For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, assume that the matrix of generation probabilities, γ , is symmetric. Consider the discrete-time dynamic system described by*

$$\mathbf{p}(t+1) = \mathbf{p}(t)\mathbf{P}, \quad (5.18)$$

where \mathbf{P} is the matrix of transition probabilities for Γ , and $\mathbf{p}(t)$ is an n -dimensional row vector at time t .

(1) *This discrete-time dynamic system has an invariant manifold. The manifold is the set of vectors*

$$\mathbf{p} \text{ with components } p_i(t) > 0, \quad 1 \leq i \leq n, \text{ and } \sum_{i=1}^n p_i(t) = 1.$$

(2) *The manifold has an equilibrium for these dynamics, $\boldsymbol{\pi}$, with components π_i satisfying (4.2).*

(3) *The function*

$$V(\mathbf{p}(t)) = - \sum_{i=1}^n \varphi_i \ln \left(\frac{p_i(t)}{\varphi_i} \right), \quad (5.19)$$

where φ_i satisfies (4.6), is a Lyapunov function that establishes global asymptotic stability of the dynamic system (5.18) with respect to the manifold.

Proof. See Appendix. □

Another important quantity of an ergodic SEGS is the expected amount of time to reach the fittest cell, given a starting cell. We will make use of the following related definitions, which are common to the theory of Markov chains [20].

Definition 11. The *return time* T_j to cell x_j is

$$T_j = \inf\{t \geq 1 \mid \mathcal{X}(t) = x_j\}, \quad (5.20)$$

where $T_j = \infty$ if $\mathcal{X}(t) \neq x_j$ for all $t \geq 1$.

The *hitting time* of x_j is taken to be

$$S_j = \begin{cases} T_j, & \text{if } \mathcal{X}(0) \neq x_j, \\ 0, & \text{if } \mathcal{X}(0) = x_j. \end{cases} \quad (5.21)$$

The *mean hitting time* to x_j given an initial cell x_i is defined as

$$\sigma_{ij} = \mathbb{E}[S_j \mid \mathcal{X}(0) = x_i] = \begin{cases} \sum_{k=1}^n P_{ik} (\sigma_{kj} + 1), & \forall i \neq j, \\ 0, & i = j, \end{cases} \quad (5.22)$$

and we take $\boldsymbol{\sigma}_j = [\sigma_{1j} \ \sigma_{2j} \ \dots \ \sigma_{nj}]^T$.

If we let $\mathbf{1} = [1 \ 1 \ \dots \ 1]^T$ and \mathbf{D}_j be a diagonal matrix with ones on the diagonal except one zero at position (j, j) , then

$$\boldsymbol{\sigma}_j = \mathbf{D}_j(\mathbf{P}\boldsymbol{\sigma}_j + \mathbf{1}). \quad (5.23)$$

Alternatively,

$$\boldsymbol{\sigma}_j = (\mathbf{I} - \mathbf{D}_j\mathbf{P})^{-1}\mathbf{D}_j\mathbf{1}, \quad (5.24)$$

where \mathbf{I} is the $n \times n$ identity matrix.

Theorem 12. For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, assume that there exists a unique index I such that $F(x_i)$ is maximized for $i = I$. Then for all $i \neq I$,

1. $\lim_{N \rightarrow \infty} \sigma_{iI}$ exists, and
2. σ_{iI} is a strictly decreasing function of N .

Proof. See Appendix. □

Hence, a trade-off exists between resilient and opportunistic behavior of the SEGS, and the expected hitting time of the optimizer, with the trade-off controlled by the level of selectivity, N . That is, increasing N reduces the mean hitting time to the fittest cell but also decreases resiliency and opportunism.

Chapter 6

Illustrative Example of a Selective Evolutionary Generation System

We illustrate the theory in this paper with an example in two parts.

6.1 Non-symmetric Matrix of Generation Probabilities

Consider the evolutionary generation system (X, R, P, G) , where

- $X = \{x_1, x_2\}$,
- $R = \{r_1, r_2\}$,
- $P(r_1) = p, P(r_2) = 1 - p, p \neq 0$,
- $G(x_1, r_1) = x_2, G(x_1, r_2) = x_1, G(x_2, r_1) = x_2$, and $G(x_2, r_2) = x_1$ (see Fig. 6.1).

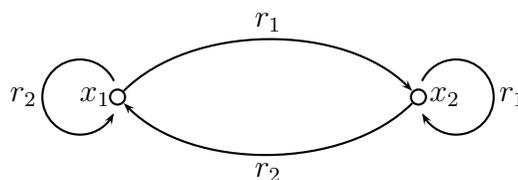


Figure 6.1: The directed graph of the example in Chapter 6.1.

The matrix of generation probabilities for this evolutionary generation system is

$$\boldsymbol{\gamma} = \begin{bmatrix} 1 - \gamma_{12} & \gamma_{12} \\ \gamma_{21} & 1 - \gamma_{21} \end{bmatrix} = \begin{bmatrix} 1 - p & p \\ 1 - p & p \end{bmatrix}. \quad (6.1)$$

Let $F(x_1) = f_1$ and $F(x_2) = f_2$. Let N be a finite level of selectivity. Utilizing (5.7), the matrix of transition probabilities for the selective evolutionary generation system is

$$\mathbf{P} = \frac{1}{f_1^N + f_2^N} \begin{bmatrix} f_1^N + (1 - \gamma_{12})f_2^N & \gamma_{12}f_2^N \\ \gamma_{21}f_1^N & (1 - \gamma_{21})f_1^N + f_2^N \end{bmatrix}. \quad (6.2)$$

This SEGS is both aperiodic and positive recurrent, and hence, ergodic.

The stationary distribution of the SEGS can be computed to be

$$\boldsymbol{\pi} = \frac{1}{\gamma_{21}f_1^N + \gamma_{12}f_2^N} \begin{bmatrix} \gamma_{21}f_1^N & \gamma_{12}f_2^N \end{bmatrix}. \quad (6.3)$$

Note that the ratio of the stationary probabilities of the two cells is

$$\frac{\pi_1}{\pi_2} = \frac{\gamma_{21}f_1^N}{\gamma_{12}f_2^N} = \frac{(1 - p)f_1^N}{pf_2^N}. \quad (6.4)$$

Since this ratio is not equal to

$$\left(\frac{f_1}{f_2} \right)^N, \quad (6.5)$$

the SEGS does not behave rationally except for $p = 0.5$. Moreover, note that for $p = 0.5$, the matrix of generation probabilities (6.1) is symmetric. Hence, this example illustrates that asymmetry of the matrix of generation probabilities may lead to behavior that is not rational. As a result, it is possible that, for small N , the most fit cell is not the most probable cell at steady-state. For instance, take $p = 0.1$, $N = 1$, $f_1 = 1$ and $f_2 = 2$. We obtain

$$\pi_1 = \frac{0.9}{0.9 + 0.2} \approx 0.818, \quad (6.6)$$

and

$$\pi_2 = \frac{0.2}{0.9 + 0.2} \approx 0.182. \quad (6.7)$$

This is why rationality is desired for optimization.

6.2 Symmetric Matrix of Generation Probabilities

Consider the evolutionary generation system (X, R, P, G) , where

- $X = \{x_1, x_2\}$,
- $R = \{r_1, r_2\}$,
- $P(r_1) = p, P(r_2) = 1 - p, p \neq 0$,
- $G(x_1, r_1) = x_2, G(x_1, r_2) = x_1, G(x_2, r_1) = x_1, \text{ and } G(x_2, r_2) = x_2$ (see Fig. 6.2).

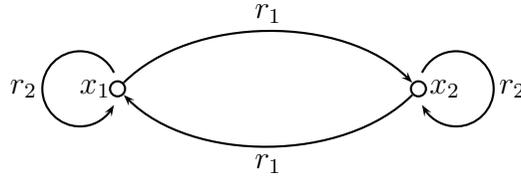


Figure 6.2: The directed graph of the example in Chapter 6.2.

The matrix of generation probabilities for this evolutionary generation system is

$$\gamma = \begin{bmatrix} 1 - \gamma_{12} & \gamma_{12} \\ \gamma_{21} & 1 - \gamma_{21} \end{bmatrix} = \begin{bmatrix} 1 - p & p \\ p & 1 - p \end{bmatrix}. \quad (6.8)$$

Let $F(x_1) = f_1$ and $F(x_2) = f_2$. Let N be a finite level of selectivity. Utilizing (5.7), the matrix of transition probabilities for the selective evolutionary generation system is

$$\mathbf{P} = \frac{1}{f_1^N + f_2^N} \begin{bmatrix} f_1^N + (1 - \gamma_{12})f_2^N & \gamma_{12}f_2^N \\ \gamma_{21}f_1^N & (1 - \gamma_{21})f_1^N + f_2^N \end{bmatrix}. \quad (6.9)$$

This SEGS is both aperiodic and positive recurrent, and hence, ergodic.

The stationary distribution of the SEGS can be computed to be

$$\boldsymbol{\pi} = \frac{1}{\gamma_{21}f_1^N + \gamma_{12}f_2^N} \begin{bmatrix} \gamma_{21}f_1^N & \gamma_{12}f_2^N \end{bmatrix}. \quad (6.10)$$

The example has been constructed such that $\gamma_{12} = \gamma_{21} = p$, and so this ergodic SEGS is rational.

Taking partial derivatives, the extrinsic and intrinsic resiliency equations of the two cells of the SEGS are

$$\rho_{11} = \frac{N\gamma_{21}f_1^{N-1}}{\gamma_{21}f_1^N + \gamma_{12}f_2^N} \left(1 - \frac{\gamma_{21}f_1^N}{\gamma_{21}f_1^N + \gamma_{12}f_2^N} \right), \quad (6.11)$$

$$= \frac{N\pi_1(1 - \pi_1)}{f_1} \neq 0, \quad (6.12)$$

$$\rho_{12} = \frac{-\gamma_{21}f_1^N}{(\gamma_{21}f_1^N + \gamma_{12}f_2^N)^2} N\gamma_{12}f_2^{N-1}, \quad (6.13)$$

$$= \frac{-N\pi_1\pi_2}{f_2} \neq 0, \quad (6.14)$$

$$\rho_{21} = \frac{-\gamma_{12}f_2^N}{(\gamma_{21}f_1^N + \gamma_{12}f_2^N)^2} N\gamma_{21}f_1^{N-1}, \quad (6.15)$$

$$= \frac{-N\pi_2\pi_1}{f_1} \neq 0, \quad (6.16)$$

$$\rho_{22} = \frac{N\gamma_{12}f_2^{N-1}}{\gamma_{21}f_1^N + \gamma_{12}f_2^N} \left(1 - \frac{\gamma_{12}f_2^N}{\gamma_{21}f_1^N + \gamma_{12}f_2^N} \right), \quad (6.17)$$

$$= \frac{N\pi_2(1 - \pi_2)}{f_2} \neq 0. \quad (6.18)$$

These equations match the theoretical results stated previously. If γ were not symmetric, this SEGS still happens to be resilient.

If $f_1 > f_2$, the mean hitting time to x_1 is

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 \\ \frac{f_1^N + f_2^N}{\gamma_{21}f_1^N} \end{bmatrix}, \quad (6.19)$$

and if $f_2 > f_1$, the mean hitting time to x_2 is

$$\sigma_2 = \begin{bmatrix} \frac{f_1^N + f_2^N}{\gamma_{12} f_2^N} \\ 0 \end{bmatrix}. \quad (6.20)$$

In the limit as N approaches infinity, we have

$$\lim_{N \rightarrow \infty} \sigma_1 = \begin{bmatrix} 0 \\ \frac{1}{\gamma_{21}} \end{bmatrix}, \quad (6.21)$$

and

$$\lim_{N \rightarrow \infty} \sigma_2 = \begin{bmatrix} \frac{1}{\gamma_{12}} \\ 0 \end{bmatrix}, \quad (6.22)$$

as expected from the proof of Theorem 12.

Chapter 7

Evolution of Flapping Wing Gaits

One possible application of selective evolutionary generation systems is the on-line selection of flapping wing gaits during flight. This application requires resilient and opportunistic optimization because mission phase transitions may change the fitness of the current flapping gait. For instance, a micro air vehicle may scout a target by favoring a hovering form of flapping flight, engage the target after increasing the fitness of descending flapping gaits, and then quickly escape after deeming ascending gaits to be the most fit. Wind fluctuations within each mission phase are another example of possible fitness perturbations.

Current optimization of low Reynolds number flapping gaits requires multiple iterations of computationally expensive three dimensional flow simulations, on multiple nodes taking days, or even weeks, to complete [45]. Moreover, these simulations depend on flow model physics that is not well understood. Thus, there is a need for a computationally inexpensive, model-independent, resilient, opportunistic, global, and on-line selection technique for flapping wing flight.

There are examples of flapping gait evolution in the literature [46–49]. The results presented in these works are either complicated by hardware-specific interactions or derived from aerodynamic and hardware models with inaccurate assumptions (e.g., steady fluid flow) for simplicity. The tolerance to fitness function perturbations is also not examined. Our contributions in this area are unique because we achieve resilient and opportunistic flapping gaits without significant

computation.

7.1 Surrogate Model

The following example applies the theory developed in this paper to a validated model that approximates the real-world physics of flapping flight. The model outputs a scalar for every acceptable input vector, and this scalar output makes it easy to discuss and verify claims of resiliency and opportunism for a realistic application.

The surrogate model for hovering flight [50] predicts a lift coefficient, C_L , for a prescribed flapping motion with various input kinematic parameters. This flapping motion is described by

$$h(t) = h_a(t) \sin(\omega t), \quad (7.1)$$

$$\alpha(t) = 90 - \alpha_a(t) \sin(\omega t + \phi_\alpha(t)), \quad (7.2)$$

where $h_a(t) \in [1, 2]$ and $\alpha_a(t) \in [45, 80]$ are the piecewise-constant amplitudes of flapping stroke height and pitch respectively, ω is a frequency that depends on h_a and a constant Reynolds number of 100, and $\phi_\alpha(t) \in [60, 120]$ is the piecewise-constant phase shift angle for flapping pitch. The flapping motion described in (7.1)–(7.2) leads to the computation of a lift coefficient, C_L , through the surrogate model. Hence, the hovering flapping flight problem: given a time history of the target lift coefficient, $C_{L_{des}}(t)$, determine suitable time-varying flapping wing kinematic parameters that meet the target.

We utilize the following evolutionary generation system, (X, R, P, G) .

- The set of cells, X , is the set of ordered triples $(h_a(t), \alpha_a(t), \phi_\alpha(t))$, where

$$h_a(t) \in \{1, 1.1, 1.2, \dots, 1.9, 2\}, \quad (7.3)$$

$$\alpha_a(t) \in \{45, 46, 47, \dots, 79, 80\}, \quad (7.4)$$

$$\phi_\alpha(t) \in \{60, 61, 62, \dots, 119, 120\}. \quad (7.5)$$

- The set of resources, R , is the set $\{r_1, r_2, r_3, r_4, r_5, r_6\}$, with $r_i = \mathbf{e}_i$, $1 \leq i \leq 6$ (where \mathbf{e}_i are the standard basis vectors for \mathbb{R}^6).
- The probability mass function on R , P , is the discrete uniform distribution.
- The generation function, G , applied to X as

$$G((h_a(t), \alpha_a(t), \phi_\alpha(t)), r_i), 1 \leq i \leq 6, \quad (7.6)$$

is the triple given by

$$\left\{ \begin{array}{l} \left[\begin{array}{cccccc} 0.1 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] r_i + \begin{bmatrix} h_a(t) \\ \alpha_a(t) \\ \phi_\alpha(t) \end{bmatrix}, \\ \text{if } 1 < h_a(t) < 2, 45 < \alpha_a(t) < 80, \\ 60 < \phi_\alpha(t) < 120, \\ (h_a(t), \alpha_a(t), \phi_\alpha(t)), \text{ otherwise.} \end{array} \right. \quad (7.7)$$

Since the objective is for $C_L(t)$ to track $C_{L_{des}}(t)$, we use the fitness function

$$F(h_a(t), \alpha_a(t), \phi_\alpha(t)) = \exp\left(-\left(K_f (C_{L_{des}}(t) - C_L(t))\right)^2\right), \quad (7.8)$$

where

$$K_f = 10, \quad (7.9)$$

and

$$C_L(t) = C_L(h_a(t), \alpha_a(t), \phi_\alpha(t)) \quad (7.10)$$

is the output of the surrogate model. Note that the fitness function in (7.8) has the following properties.

- Akin to a membership function, the fitness function is normalized so that a fitness between 0 and 1 is achieved depending on how well the model output matches the desired output. A fitness of 1 represents a perfect output match, whereas a fitness of 0 signifies a poor match.
- The fitness function utilizes a gain parameter, K_f , which indicates how dissimilar the desired output and a high-fitness true output are tolerated to be. Larger gains indicate that the SEGS is more permissive of poor matches. The gain parameter is also related to the level of selectivity, N , because the latter is always used as an exponent of fitness. Hence, in the above fitness function, K_f plays a similar role to N .
- Corresponding to the above, it can be shown that the fitness function is proportional to a Gaussian probability density function with mean equal to the desired output, variance equal to $\frac{1}{2K_f^2}$, and a constant of proportionality equal to $\sqrt{\frac{2\pi}{K_f^2}}$.

7.2 Surrogate Model Results

A sample run of the evolution scheme when $N = 5$ is depicted in Figs. 7.1 to 7.4. A cell triple that achieves satisfactory performance is found within 1000 generations, and the scheme is resilient because it quickly finds a new triple that achieves an acceptable output when the target lift coefficient, and hence the fitness function, changes. In Figs. 7.1 to 7.3, the red vertical dashed lines indicate a generation for which the evolved flapping forward and backward motion is illustrated in Fig. 7.4.

For generations 1, 900, 1025, and 2000, the plots in Fig. 7.4 each display 10 snapshots of a 15% elliptical airfoil through a flapping half-stroke. The solid circle represents the leading edge

of the airfoil, which moves in an aircraft body-fixed reference frame with neutral position at (0,0). The arrows on the forward half-stroke plots indicate that the airfoil travels from the most rearward position to the most forward position, whereas the opposite is true for a backward half-stroke. Although the periods of the strokes vary at different generations because of the constant Reynolds number, the snapshots are taken at the same fractional period interval. Therefore, a stroke with more spacing between snapshots has a faster motion than a stroke with snapshots that are closely spaced.

Typically, the scheme averages 1 minute 18 seconds to compute the output of 1000 generations while running in MATLAB on a 2.50 GHz dual-core processor laptop with 4.00 GB of RAM and the Windows Vista operating system.

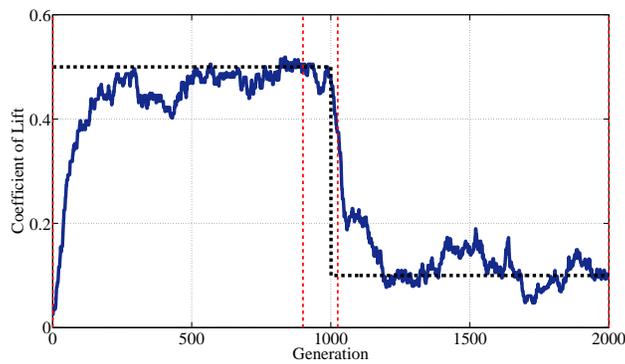


Figure 7.1: Target (dashed black) and actual (solid blue) lift coefficients per generation.

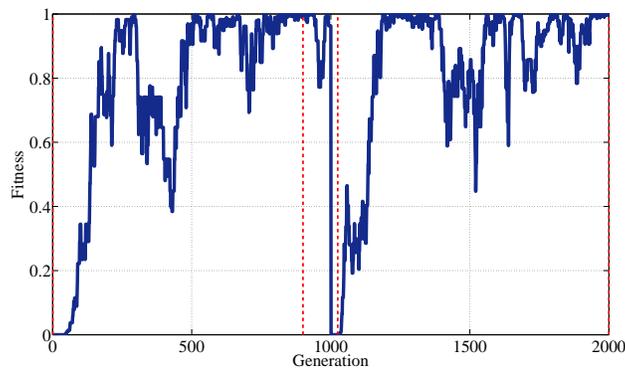


Figure 7.2: Fitness value per generation.

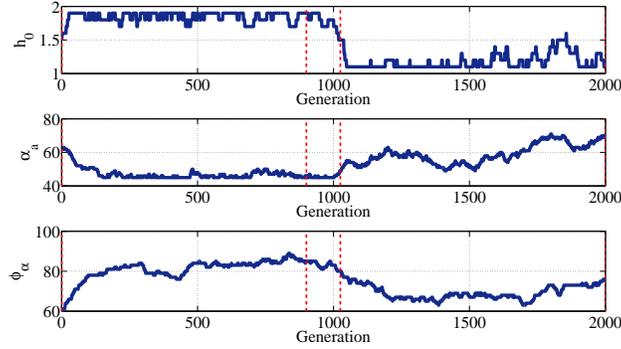


Figure 7.3: Flapping wing kinematic parameters per generation.

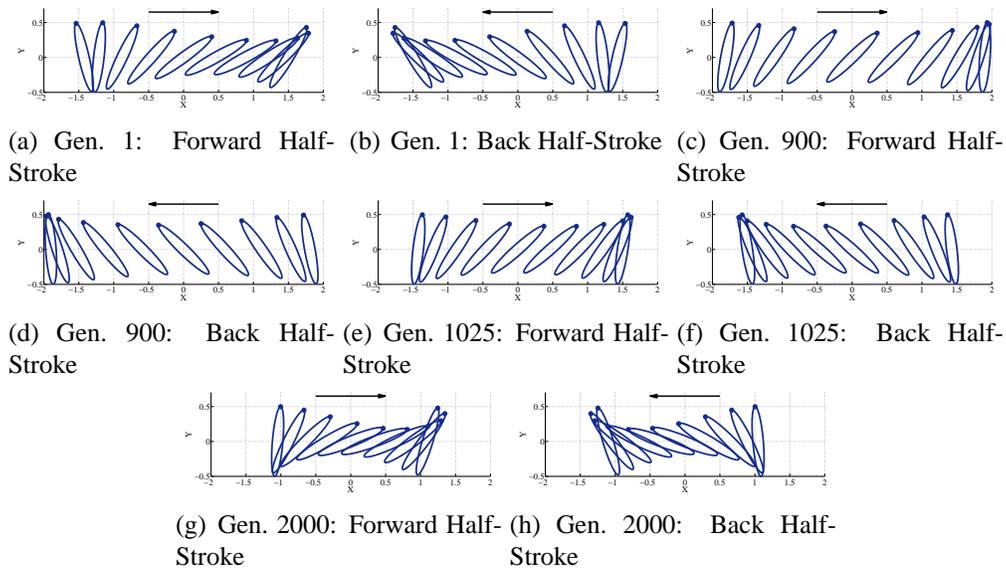


Figure 7.4: Snapshots of the forward and back half-strokes of the flapping wing sampled at the 1st, 900th, 1025th, and 2000th generations.

7.3 Theodorsen-Garrick Model

This example utilizes the developed theory and a model embodied by unsteady flow equations to consider the standard reference-tracking problem in control systems within the context of flapping flight. The work here is different from [51–57] in that we evolve flapping wing parameters for trajectory tracking, instead of taking a control-theoretic approach. The chosen model outputs forces from which trajectories can be computed, and these trajectories are then analyzed by a SEGS. This approach, and the other differences from the previous example (forward motion of the flapping

wing, incorporation of actual physics, and an addition to the literature), are a legitimate reason to include the application here.

The Theodorsen-Garrick model [58, 59] predicts the lift and thrust forces on a flat plate undergoing a prescribed flapping motion with various input kinematic parameters. This flapping motion is described by

$$h(t) = h_a(t) \sin(\omega(t)t + \phi_h(t)), \quad (7.11)$$

$$\alpha(t) = \alpha_a(t) \sin(\omega(t)t + \phi_\alpha(t)), \quad (7.12)$$

where $h_a(t) \in (0, 1]$ and $\alpha_a(t) \in [-0.5, 0.5]$ are the piecewise-constant amplitudes of flapping stroke height and angle of attack respectively, $\omega(t) \in (0, 1]$ is a piecewise-constant frequency, and $\phi_h(t) \in [-0.5, 0.5]$ and $\phi_\alpha(t) \in [-0.5, 0.5]$ are the piecewise-constant phase shift angles for flapping stroke height and angle of attack, respectively. The flapping motion described in (7.11)–(7.12) leads to the computation of lift and thrust forces through the equations stated in [59]. These forces determine the trajectory followed by the flapping wing; hence, the flapping flight motion problem: given a target trajectory (e.g., a constant altitude forward motion trajectory), find suitable flapping wing kinematic parameters that meet the target.

We utilize the following evolutionary generation system, (X, R, P, G) .

- The set of cells, X , is the set of ordered pentuples $(h_a(t), \omega(t), \phi_h(t), \alpha_a(t), \phi_\alpha(t))$, where

$$h_a(t) \in \{0.1, 0.2, 0.3, \dots, 0.9, 1\}, \quad (7.13)$$

$$\omega(t) \in \{0.05, 0.1, 0.15, \dots, 0.95, 1\}, \quad (7.14)$$

$$\phi_h(t) \in \{-0.5, -0.45, -0.4, \dots, 0.45, 0.5\}, \quad (7.15)$$

$$\alpha_a(t) \in \{-0.5, -0.45, -0.4, \dots, 0.45, 0.5\}, \quad (7.16)$$

$$\phi_\alpha(t) \in \{-0.5, -0.45, -0.4, \dots, 0.45, 0.5\}. \quad (7.17)$$

- The set of resources, R , is the set $\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}\}$, with $r_i = \mathbf{e}_i$, $1 \leq i \leq 10$ (where \mathbf{e}_i are the standard basis vectors for \mathbb{R}^{10}).
- The probability mass function on R , P , is the discrete uniform distribution.
- The generation function, G , applied to X as

$$G((h_a(t), \omega(t), \phi_h(t), \alpha_a(t), \phi_\alpha(t)), r_i),$$

$$1 \leq i \leq 10, \tag{7.18}$$

is the pentuple given by

$$\left\{ \begin{array}{l}
 \begin{array}{c}
 h_a(t) \\
 \omega(t) \\
 \phi_h(t) \\
 \alpha_a(t) \\
 \phi_\alpha(t)
 \end{array} \\
 1 \times 10^{-2} A r_i + \end{array} \right. , \text{ where } A = \begin{bmatrix}
 10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 5 & -5 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5
 \end{bmatrix} , \tag{7.19}$$

$$\left\{ \begin{array}{l}
 \text{if } 0.1 < h_a(t) < 1, 0.05 < \omega(t) < 1, \\
 -0.5 < \phi_h(t) < 0.5, -0.5 < \alpha_a(t) < 0.5, \\
 -0.5 < \phi_\alpha(t) < 0.5, \\
 (h_a(t), \omega(t), \phi_h(t), \alpha_a(t), \phi_\alpha(t)), \text{ otherwise.}
 \end{array} \right.$$

The flapping wing parameters evolved by the SEGS are inputs for the Theodorsen-Garrick model, which outputs lift $L(\tau)$ and time-averaged-thrust $T(\tau)$ over time τ . These forces are in

turn inputs for the following double-integrator, unit-mass wing trajectory dynamics,

$$\begin{bmatrix} \dot{x}(\tau) \\ \dot{y}(\tau) \\ \dot{v}_x(\tau) \\ \dot{v}_y(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(\tau) \\ y(\tau) \\ v_x(\tau) \\ v_y(\tau) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} L(\tau) \\ T(\tau) \end{bmatrix}, \quad (7.20)$$

where $(x(\tau), y(\tau))$ is the trajectory of the center of mass of the flapping wing. This trajectory is sampled ν times, yielding $(x(k), y(k)), 1 \leq k \leq \nu$. For each $x(k)$, the target $y_{des}(k)$ is computed.

Let

$$AvgDistance(t) = \frac{\sum_{k=1}^{\nu} |y_{des}(k) - y(k)|}{\nu} \quad (7.21)$$

be the mean difference between the target and current trajectories. Since the objective is to track the target, we use the following fitness function for the SEGS,

$$F(h_a(t), \omega(t), \phi_h(t), \alpha_a(t), \phi_\alpha(t)) = \exp\left(- (0.1 AvgDistance(t))^2\right). \quad (7.22)$$

7.4 Theodorsen-Garrick Model Results

A sample initial trajectory together with a trajectory obtained from that sample after 200 generations with $N = 5$ are plotted in Fig. 7.5, where the trajectories are depicted over the same period of time. The figure shows that the evolved kinematic parameters reduce altitude excursions away from the target trajectory by a factor of four while utilizing roughly the same amount of time-averaged-thrust that was specified by the initial set of kinematic parameters. Moreover, the average evolved trajectory tracks the constant altitude desired trajectory, while the average initial trajectory does not.

The scheme requires, on average, 2 minutes 34 seconds to compute the output of 200 genera-

tions while running in MATLAB on a 2.50 GHz dual-core processor laptop with 4.00 GB of RAM and the Windows Vista operating system.

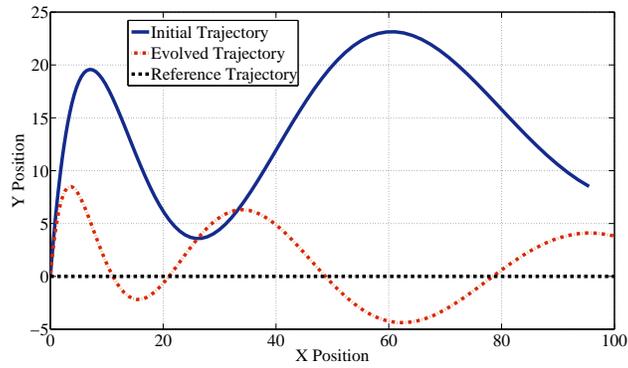


Figure 7.5: Target trajectory (dashed black), initial trajectory (solid blue) and the 200th evolved (dashed-dotted red) trajectory.

Chapter 8

Conclusions and Future Work

This paper has proposed a novel on-line global optimization strategy by demonstrating and utilizing the fact that the desirable characteristics of resiliency and opportunism are guaranteed by rational behavior. The ratio of the stationary probability of the optimizer of a fitness function to any other element's stationary probability is given by

$$\frac{\pi_I}{\pi_j} = \left(\frac{F(x_I)}{F(x_j)} \right)^N, \quad 1 \leq j \leq n, \quad (8.1)$$

where $F(x_I) > F(x_i)$ for all i implies that cell x_I is the most likely. In the limit as N approaches infinity, π_I approaches 1, and standard optimization is recovered. The canonical genetic algorithm with fitness proportional selection and the (1+1)-ES strategy are particular cases of the proposed scheme.

Although rational behavior suggests optimization decisions that are based on global knowledge, this paper proves that rationality may be achieved through a sequence of local decisions,

$$\mathcal{X}(t+1) = \text{Select}(\mathcal{X}(t), G(\mathcal{X}(t), \mathcal{R}(t)), N), \quad (8.2)$$

that require limited knowledge of the objective function. Thus, each step of the proposed scheme is also computationally inexpensive.

Resiliency and opportunism are achieved at the expense of the mean hitting time to the optimizer, and the trade-off is managed through the level of selectivity N . The resiliency of a SEGS is a conserved quantity, and any improvements to the resiliency of a particular element decreases the resiliency of other elements.

We have utilized our technique to successfully develop resilient hovering and forward-motion flapping wing gaits without expending significant computation effort.

The work here may be extended to the case of multi-objective (Pareto) optimization through a suitable definition of the fitness function. Future work includes understanding the origins of the Lyapunov function that characterizes a SEGS, investigating the implications of time-reversibility, and exploring parallel computing implementations of this work. We will also seek a conservation law for resiliency and mean hitting time.

Appendix A

Proofs

Theorem 1.

Proof. To show that (4.2) implies Markov chain rational behavior, consider the ratio of any π_i to π_j , $i \neq j$, where each satisfies (4.2). Equation (4.1) follows immediately.

To show that Markov chain rational behavior implies (4.2), we begin with

$$\sum_{k=1}^n \pi_k = 1.$$

Dividing both sides of the equation by π_i , we obtain

$$\sum_{k=1}^n \frac{\pi_k}{\pi_i} = \frac{1}{\pi_i}, \quad 1 \leq i \leq n,$$

which, using (4.1), yields

$$\sum_{k=1}^n \left(\frac{F(x_k)}{F(x_i)} \right)^N = \frac{1}{\pi_i}, \quad 1 \leq i \leq n.$$

Multiplying by $F(x_i)^N$ and solving for π_i yields (4.2), which completes the proof. \square

Theorem 2.

Proof. We use the method of Karush-Kuhn-Tucker (KKT) multipliers to solve the optimization

problem

$$\min_{\pi_1, \dots, \pi_n} \Phi(\boldsymbol{\pi}) = - \sum_{i=1}^n F(x_i)^N \ln(\pi_i),$$

subject to

$$\begin{aligned} \sum_{i=1}^n \pi_i - 1 &= 0, \\ -\pi_i &< 0, \quad 1 \leq i \leq n. \end{aligned}$$

Let $L(\pi_1, \dots, \pi_n, \lambda, \mu_1, \dots, \mu_n) =$

$$- \sum_{i=1}^n F(x_i)^N \ln(\pi_i) + \lambda \left(\sum_{i=1}^n \pi_i - 1 \right) - \sum_{i=1}^n \mu_i \pi_i.$$

The KKT necessary conditions for optimality are

$$\frac{-F(x_i)^N}{\pi_i} + \lambda - \mu_i = 0, \quad 1 \leq i \leq n,$$

$$\sum_{i=1}^n \pi_i - 1 = 0,$$

$$-\pi_i < 0, \quad 1 \leq i \leq n,$$

$$\lambda \geq 0,$$

$$\mu_i \geq 0, \quad 1 \leq i \leq n,$$

$$\lambda \left(\sum_{i=1}^n \pi_i - 1 \right) = 0,$$

$$\mu_i \pi_i = 0, \quad 1 \leq i \leq n.$$

The first necessary condition becomes

$$-F(x_i)^N + \lambda \pi_i - \mu_i \pi_i = 0, \quad 1 \leq i \leq n.$$

Since $\mu_i \pi_i = 0$ for all i , we obtain

$$-F(x_i)^N + \lambda \pi_i = 0, \quad 1 \leq i \leq n.$$

Next, the constraint $\pi_i > 0$ for all i and the positive nature of $F(x_i)^N$ imply that $\lambda \neq 0$. Therefore,

$$\pi_i = \frac{F(x_i)^N}{\lambda}, \quad 1 \leq i \leq n.$$

$$\sum_{i=1}^n \pi_i = \sum_{i=1}^n \frac{F(x_i)^N}{\lambda}, \quad 1 \leq i \leq n.$$

Since $\sum_{i=1}^n \pi_i = 1$, we find that

$$\lambda = \sum_{i=1}^n F(x_i)^N,$$

and hence,

$$\pi_i = \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N}, \quad 1 \leq i \leq n.$$

Thus, the stationary distribution in (4.2) satisfies the first order necessary conditions for optimality.

Moreover, we have

$$\frac{\partial^2 \Phi(\boldsymbol{\pi})}{\partial \pi_j \partial \pi_i} = 0 \text{ for } j \neq i,$$

$$\frac{\partial^2 \Phi(\boldsymbol{\pi})}{\partial \pi_i^2} = \frac{F(x_i)^N}{\pi_i^2} > 0.$$

Hence, the optimization problem has a strictly convex cost function and linear constraints. Thus, the solution of the first order necessary conditions is the global optimizer, which completes the proof. □

Theorem 3.

Proof. Similar to Theorem 2, we can use the method of Karush-Kuhn-Tucker (KKT) multipliers

to solve the following optimization problem for arbitrary $\varphi \in \mathbb{D}_n$:

$$\min_{\boldsymbol{\pi} \in \mathbb{D}_n} - \sum_{i=1}^n \varphi_i \ln(\pi_i),$$

which is equivalent to

$$\min_{\pi_1, \dots, \pi_n} \Phi(\boldsymbol{\pi}) = - \sum_{i=1}^n \varphi_i \ln(\pi_i),$$

subject to

$$\begin{aligned} \sum_{i=1}^n \pi_i - 1 &= 0, \\ -\pi_i &< 0, \quad 1 \leq i \leq n. \end{aligned}$$

This is a scaled version of Theorem 2, and therefore the remainder of the proof is omitted. \square

Theorem 4.

Proof. We prove both parts of this theorem directly. Consider that

$$\begin{aligned} \rho_{ij} \Big|_{N=0} &= \frac{-N\pi_i\pi_j}{F(x_j)} \Big|_{N=0}, \\ &= \frac{-N}{F(x_j)} \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N} \frac{F(x_j)^N}{\sum_{k=1}^n F(x_k)^N} \Big|_{N=0}. \end{aligned}$$

By substitution, $\rho_{ij} \Big|_{N=0}$ is 0. Similarly,

$$\begin{aligned} \rho_{ii} \Big|_{N=0} &= \frac{N\pi_i(1-\pi_i)}{F(x_i)} \Big|_{N=0}, \\ &= \frac{N}{F(x_i)} \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N} \left(1 - \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N} \right) \Big|_{N=0}. \end{aligned}$$

By substitution, $\rho_{ii} \Big|_{N=0}$ is also 0.

For the second part of the theorem, we need the following lemma that can be easily proven

using L'Hôpital's rule.

Lemma 2. *Let $0 < \alpha < 1$. Then $\lim_{N \rightarrow \infty} N\alpha^N = 0$.*

Let I be the index for which $F(x_i)$ is maximized, and assume that I is unique. Then,

$$\lim_{N \rightarrow \infty} \frac{F(x_j)^N}{F(x_I)^N} = 0, \quad \forall j \neq I, \text{ and}$$

$$\lim_{N \rightarrow \infty} \sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N} = 1.$$

Consider that

$$\begin{aligned} \lim_{N \rightarrow \infty} \rho_{ij} &= \lim_{N \rightarrow 0} \frac{-N\pi_i\pi_j}{F(x_j)}, \\ &= \lim_{N \rightarrow \infty} \frac{-N}{F(x_j)} \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N} \frac{F(x_j)^N}{\sum_{k=1}^n F(x_k)^N}, \\ &= \lim_{N \rightarrow \infty} \frac{-N}{F(x_j)} \frac{\frac{F(x_i)^N}{F(x_I)^N}}{\sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N}} \frac{\frac{F(x_j)^N}{F(x_I)^N}}{\sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N}}. \end{aligned}$$

Now for all $i \neq j$, where $i \neq I$ and $j \neq I$, the application of Lemma 2 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim_{N \rightarrow \infty} \rho_{ij} = 0$.

If $i = I \neq j$, then the application of Lemma 2 with $\alpha = \frac{F(x_j)}{F(x_I)}$ implies that $\lim_{N \rightarrow \infty} \rho_{ij} = 0$.

Lastly, if $i \neq j = I$, then the application of Lemma 2 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim_{N \rightarrow \infty} \rho_{ij} = 0$.

Thus, for all i and j , $\lim_{N \rightarrow \infty} \rho_{ij} = 0$.

Similarly,

$$\begin{aligned} \lim_{N \rightarrow \infty} \rho_{ii} &= \lim_{N \rightarrow 0} \frac{N\pi_i(1-\pi_i)}{F(x_i)}, \\ &= \lim_{N \rightarrow \infty} \frac{N}{F(x_i)} \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N} \left(1 - \frac{F(x_i)^N}{\sum_{k=1}^n F(x_k)^N} \right), \end{aligned}$$

$$= \lim_{N \rightarrow \infty} \frac{N}{F(x_i)} \frac{\frac{F(x_i)^N}{F(x_I)^N}}{\sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N}} \frac{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{F(x_k)^N}{F(x_I)^N}}{\sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N}}.$$

If $i \neq I$, then the application of Lemma 2 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim_{N \rightarrow \infty} \rho_{ii} = 0$.

If $i = I$, then we have

$$\lim_{N \rightarrow \infty} \rho_{ii} = \lim_{N \rightarrow \infty} \frac{N}{F(x_I)} \frac{\frac{F(x_I)^N}{F(x_I)^N}}{\sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N}} \frac{\sum_{\substack{k=1 \\ k \neq I}}^n \frac{F(x_k)^N}{F(x_I)^N}}{\sum_{k=1}^n \frac{F(x_k)^N}{F(x_I)^N}}.$$

The application of Lemma 2 with $\alpha = \frac{F(x_k)}{F(x_I)}$ a total of $n - 1$ times implies that $\lim_{N \rightarrow \infty} \rho_{ii} = 0$.

Thus, for all i , $\lim_{N \rightarrow \infty} \rho_{ii} = 0$. This completes the proof. \square

Theorem 5.

Proof. To show that rational behavior implies that the time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) is resilient and opportunistic, consider (4.16) and (4.17), which hold because the stationary distribution π has the closed form expression (4.2). By Definition 5, $\pi_i > 0 \forall i$ since the Markov chain is ergodic, $N > 0$ since the Markov chain is selective, and $F(x_i) > 0 \forall i$ since the fitness function is positive. Hence, $\rho_{ij} \neq 0 \forall i$ and j , and (X, \mathbf{P}) is resilient and opportunistic. This completes the proof. \square

Theorem 6.

Proof. To show that ergodicity is a necessary condition for the time-homogeneous, irreducible, ergodic Markov chain (X, \mathbf{P}) to be resilient and opportunistic, suppose that the chain is not ergodic. Then the chain is either not positive recurrent (i.e., it is null recurrent or transient) or it is periodic. If the chain is not positive recurrent, then there exists a state, x_i , with zero stationary probability. Suppose now that the fitness function is perturbed such that the fitness of this state, $F(x_i)$, becomes the optimal fitness value. Since the stationary probability of x_i is zero, state x_i is never visited, and therefore never considered as the optimizer. We have $\rho_{ii} = \partial \pi_i / \partial F(x_i) = 0$, and hence (X, \mathbf{P}) is

not resilient or opportunistic. If the chain is periodic, then the stationary probability distribution does not exist, and resiliency and opportunism are not defined. This completes the proof. \square

Lemma 1.

Proof. We prove the claim directly. Using (5.3), we have

$$\begin{aligned} \sum_{j=1}^n \gamma_{ij} &= \sum_{j=1}^n \sum_{k=1}^m \delta_{ijk} p_k, \\ &= \sum_{k=1}^m \sum_{j=1}^n \delta_{ijk} p_k, \\ &= \sum_{k=1}^m p_k \sum_{j=1}^n \delta_{ijk}. \end{aligned}$$

Now, $\sum_{j=1}^n \delta_{ijk} = 1$ because cell x_i and resource r_k generate a unique cell $G(x_i, r_k)$. Therefore,

$$\begin{aligned} \sum_{j=1}^n \gamma_{ij} &= \sum_{k=1}^m p_k \cdot 1, \\ &= \sum_{k=1}^m p_k = 1. \end{aligned}$$

This completes the proof. \square

Theorem 7.

Proof. We prove the claim directly. Using (5.7), we have

$$\begin{aligned} \sum_{j=1}^n P_{ij} &= \sum_{\substack{j=1 \\ j \neq i}}^n P_{ij} + P_{ii}, \\ &= \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_i)}{F(x_j)}\right)^N} \gamma_{ij} + \gamma_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \gamma_{ij}, \\ &= \sum_{\substack{j=1 \\ j \neq i}}^n \left(\frac{1}{1 + \left(\frac{F(x_i)}{F(x_j)}\right)^N} + \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \right) \gamma_{ij} + \gamma_{ii}, \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{j=1 \\ j \neq i}}^n \gamma_{ij} + \gamma_{ii}, \\
&= \sum_{j=1}^n \gamma_{ij} = 1.
\end{aligned}$$

This completes the proof. □

Theorem 8.

Proof. This is a direct proof. We begin by noting that

$$\begin{aligned}
\lim_{N \rightarrow \infty} P_{ij} &= \lim_{N \rightarrow \infty} \frac{1}{1 + \left(\frac{F(x_i)}{F(x_j)}\right)^N} \gamma_{ij}, \\
&= \begin{cases} \gamma_{ij}, & \text{if } F(x_i) < F(x_j), \\ 0, & \text{if } F(x_i) > F(x_j), \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{N \rightarrow \infty} P_{ii} &= \lim_{N \rightarrow \infty} \left(\gamma_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \gamma_{ij} \right), \\
&= \gamma_{ii} + \sum_{\substack{j=1 \\ j \neq i \\ F(x_i) > F(x_j)}}^n \gamma_{ij}, \\
&= 1 - \sum_{\substack{j=1 \\ j \neq i \\ F(x_i) < F(x_j)}}^n \gamma_{ij}.
\end{aligned}$$

Without loss of generality, assume that the cells of the SEGS are ordered according to decreasing fitness value, so that the index $I = 1$. The matrix $\lim_{N \rightarrow \infty} \mathbf{P}$ is therefore a lower triangular matrix. Furthermore, $\lim_{N \rightarrow \infty} P_{11} = 1$.

Consider the row vector $\mathbf{v} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$. The product of this row vector with the lower triangular matrix $\lim_{N \rightarrow \infty} \mathbf{P}$ is the first row of $\lim_{N \rightarrow \infty} \mathbf{P} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = \mathbf{v}$.

Therefore, the row vector $\mathbf{v} = [1 \ 0 \ \dots \ 0]$ is a left eigenvector of $\lim_{N \rightarrow \infty} \mathbf{P}$, with corresponding eigenvalue 1 (i.e., $\mathbf{v} \lim_{N \rightarrow \infty} \mathbf{P} = \mathbf{v}$). Hence, $\lim_{N \rightarrow \infty} \boldsymbol{\pi} = \mathbf{v}$, and the proof is complete. \square

Theorem 9.

Proof. We directly show that the row vector $\boldsymbol{\pi} = [\pi_1 \ \pi_2 \ \dots \ \pi_n]$, where π_i satisfies (4.2), is a left eigenvector of \mathbf{P} , the matrix of transition probabilities for Γ , with corresponding eigenvalue 1. If the matrix of generation probabilities, $\boldsymbol{\gamma}$, is symmetric, then

$$\gamma_{ij} = \gamma_{ji}, \quad 1 \leq i, j \leq n,$$

or equivalently,

$$\sum_{k=1}^m \delta_{ijk} p_k = \sum_{k=1}^m \delta_{jik} p_k.$$

Consider the row vector $\mathbf{v} = \boldsymbol{\pi} \mathbf{P}$. Then

$$\begin{aligned} v_j &= \sum_{i=1}^n \pi_i P_{ij}, \\ &= \sum_{\substack{i=1 \\ i \neq j}}^n \pi_i P_{ij} + \pi_j P_{jj}, \\ &= \sum_{\substack{i=1 \\ i \neq j}}^n \pi_i P_{ij} + \pi_j \left(1 - \sum_{\substack{i=1 \\ i \neq j}}^n P_{ji} \right), \\ &= \sum_{\substack{i=1 \\ i \neq j}}^n \pi_i P_{ij} + \pi_j - \sum_{\substack{i=1 \\ i \neq j}}^n \pi_j P_{ji}. \end{aligned}$$

From (4.2), (5.3), and (5.7), v_j becomes

$$\begin{aligned} &\sum_{\substack{i=1 \\ i \neq j}}^n \left(\frac{F(x_i)^N}{\sum_{a=1}^n F(x_a)^N} \frac{F(x_j)^N}{F(x_i)^N + F(x_j)^N} \sum_{k=1}^m \delta_{ijk} p_k \right) \\ &+ \pi_j \end{aligned}$$

$$- \sum_{\substack{i=1 \\ i \neq j}}^n \left(\frac{F(x_j)^N}{\sum_{a=1}^n F(x_a)^N} \frac{F(x_i)^N}{F(x_i)^N + F(x_j)^N} \sum_{k=1}^m \delta_{jik} p_k \right).$$

This reduces to π_j because γ is symmetric. Hence, $\pi = \pi P$. □

Theorem 10.

Proof. We directly show that $\pi_i P_{ij} = \pi_j P_{ji}$ for all i and j . If the matrix of generation probabilities, γ , is symmetric, then

$$\gamma_{ij} = \gamma_{ji}, \quad 1 \leq i, j \leq n,$$

or equivalently,

$$\sum_{k=1}^m \delta_{ijk} p_k = \sum_{k=1}^m \delta_{jik} p_k, \quad 1 \leq i, j \leq n.$$

Consider $\pi_i P_{ij}$. Using (4.2), (5.3), and (5.7), we obtain

$$\begin{aligned} \pi_i P_{ij} &= \frac{F(x_i)^N}{\sum_{a=1}^n F(x_a)^N} \frac{F(x_j)^N}{F(x_i)^N + F(x_j)^N} \sum_{k=1}^m \delta_{ijk} p_k, \\ &= \frac{F(x_j)^N}{\sum_{a=1}^n F(x_a)^N} \frac{F(x_i)^N}{F(x_i)^N + F(x_j)^N} \sum_{k=1}^m \delta_{jik} p_k, \\ &= \pi_j P_{ji}, \quad 1 \leq i, j \leq n, \end{aligned}$$

where the second equation uses the symmetry of γ . Hence, the Markov chain representing the stochastic dynamics of the ergodic SEGS is time-reversible. □

Theorem 11.

Proof. We use Lyapunov's Method and the LaSalle Invariance Principle [60, 61] to directly prove this theorem.

For the ergodic SEGS $\Gamma = (X, R, P, G, F)$ with a symmetric matrix of generation probabili-

ties, γ , consider the discrete-time dynamic system described by

$$\mathbf{p}(t+1) = \mathbf{p}(t)\mathbf{P},$$

where \mathbf{P} is the matrix of transition probabilities for Γ , and $\mathbf{p}(t)$ is an n -dimensional row vector at time t . Here, $\mathbf{p}(t)$ is the ergodic probability distribution over the states at time t , and therefore the components of \mathbf{p} satisfy $p_i(t) > 0$, $1 \leq i \leq n$, and $\sum_{i=1}^n p_i(t) = 1$. Since the SEGS is ergodic and irreducible, a unique equilibrium stationary distribution for these dynamics exists, $\lim_{t \rightarrow \infty} \mathbf{p}(t) = \boldsymbol{\pi}$, with components π_i satisfying (4.2).

Let us define $\mathbf{q}(t) = \mathbf{p}(t) - \boldsymbol{\pi}$, so that the transformed discrete-time dynamic system,

$$\mathbf{q}(t+1) = (\mathbf{q}(t) + \boldsymbol{\pi})\mathbf{P} - \boldsymbol{\pi},$$

has an equilibrium at the origin. The function

$$V(\mathbf{p}(t)) = -\sum_{i=1}^n \varphi_i \ln \left(\frac{p_i(t)}{\varphi_i} \right),$$

where φ_i satisfies (4.6), may be rewritten as

$$V(\mathbf{q}(t) + \boldsymbol{\pi}) = -\sum_{i=1}^n \varphi_i \ln \left(\frac{q_i(t) + \pi_i}{\varphi_i} \right).$$

We first check the value of this transformed candidate Lyapunov equation at the origin of the transformed system. We have

$$V(\mathbf{0} + \boldsymbol{\pi}) = -\sum_{i=1}^n \varphi_i \ln \left(\frac{\pi_i}{\varphi_i} \right) = -\sum_{i=1}^n \varphi_i \ln 1 = 0,$$

because $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Next, we have to show that $\forall \mathbf{q}(t) \neq \mathbf{0}$, $V(\mathbf{q}(t) + \boldsymbol{\pi}) > 0$. But this follows directly from (the

second equivalent restatement of) Theorem 3. This is because $\forall \mathbf{q}(t) \neq \mathbf{0}$,

$$V(\mathbf{q}(t) + \boldsymbol{\pi}) = - \sum_{i=1}^n \varphi_i \ln \left(\frac{p_i(t)}{\varphi_i} \right),$$

which is always positive according to the theorem.

Now consider $\Delta V = V(\mathbf{q}(t+1) + \boldsymbol{\pi}) - V(\mathbf{q}(t) + \boldsymbol{\pi})$. In the equations that follow, we assume, without loss of generality, that the fitness value of each cell of the SEGS is greater than or equal to one. (After all, if there exists an i such that $0 < F(x_i) < 1$, then it is possible to find a $K \in \mathbb{R}^+$ to scale all the fitness values upward, so that for all i , $KF(x_i) \geq 1$. Define the new fitnesses $F'(x_i) = KF(x_i)$, $1 \leq i \leq n$ and observe that the Markov chain representation of the SEGS is unchanged).

$$\begin{aligned} \Delta V &= V(\mathbf{p}(t+1)) - V(\mathbf{p}(t)), \\ &= - \sum_{j=1}^n \varphi_j \ln \left(\frac{p_j(t+1)}{\varphi_j} \right) + \sum_{j=1}^n \varphi_j \ln \left(\frac{p_j(t)}{\varphi_j} \right), \\ &= - \sum_{j=1}^n \varphi_j \ln \left(\frac{p_j(t+1)}{p_j(t)} \right), \\ &= - \sum_{j=1}^n \varphi_j \ln \left(\frac{\sum_{i=1}^n p_i(t) P_{ij}}{p_j(t)} \right), \\ &= - \sum_{j=1}^n \varphi_j \ln \left(\frac{1}{p_j(t)} \sum_{i=1}^n \frac{p_i(t) F(x_j)^N}{F(x_i)^N + F(x_j)^N} \right), \\ &= - \sum_{j=1}^n \varphi_j \ln \left(\frac{F(x_j)^N}{p_j(t)} \sum_{i=1}^n \frac{p_i(t)}{F(x_i)^N + F(x_j)^N} \right). \end{aligned}$$

Now because we have assumed, without loss of generality, that all fitnesses are greater than or equal to one, we have

$$\frac{F(x_j)^N}{p_j(t)} \sum_{i=1}^n \frac{p_i(t)}{F(x_i)^N + F(x_j)^N}$$

$$\begin{aligned}
&\geq \sum_{i=1}^n \frac{p_i(t)}{F(x_i)^N + F(x_j)^N}, \\
&\geq \sum_{i=1}^n p_i(t), \\
&\geq 1.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\Delta V &\leq - \sum_{j=1}^n \varphi_j \ln 1, \text{ or} \\
\Delta V &\leq 0.
\end{aligned}$$

That is, ΔV is negative semi-definite, as required by Lyapunov's method.

To apply LaSalle's Invariance Principle, we have to find $Q = \{\mathbf{q}(t) | \Delta V = 0\}$. Note that

$$\begin{aligned}
\Delta V &= 0, \\
&= - \sum_{j=1}^n \varphi_j \ln 1, \\
&= - \sum_{j=1}^n \varphi_j \ln \left(\sum_{i=1}^n P_{ji} \right),
\end{aligned}$$

which can be rewritten with Bayes' Rule as

$$\Delta V = - \sum_{j=1}^n \varphi_j \ln \left(\sum_{i=1}^n \frac{\pi_i}{\pi_j} P_{ij} \right).$$

We had previously shown that

$$\Delta V = - \sum_{j=1}^n \varphi_j \ln \left(\frac{\sum_{i=1}^n p_i(t) P_{ij}}{p_j(t)} \right).$$

Thus, $\Delta V = 0$ implies that $\pi_i = p_i(t)$, $1 \leq i \leq n$. But from the definition of $\mathbf{q}(t)$,

$$\pi_i = p_i(t) - q_i(t),$$

and we must have that $\Delta V = 0$ implies that $q_i(t) = 0$, $1 \leq i \leq n$. Therefore, the only solution of the transformed discrete-time dynamic system that can stay identically in Q is the trivial solution $\mathbf{q}(t) \equiv 0$. Hence, the origin is an asymptotically stable equilibrium for the transformed discrete-time dynamic system, and therefore, the function

$$V(\mathbf{p}(t)) = - \sum_{i=1}^n \varphi_i \ln \left(\frac{p_i(t)}{\varphi_i} \right),$$

is a Lyapunov function for the original system with the set of vectors \mathbf{p} with components $p_i(t) > 0$, $1 \leq i \leq n$, and $\sum_{i=1}^n p_i(t) = 1$ forming an invariant manifold. Moreover, since the Lyapunov function is radially unbounded, the equilibrium is globally asymptotically stable, as claimed. \square

Theorem 12.

Proof. We first prove directly that σ_{iI} converges to a constant value for each i as N approaches infinity, before inductively showing that the value of σ_{iI} does indeed decrease with increasing N .

We begin by noting that

$$\begin{aligned} \lim_{N \rightarrow \infty} P_{ij} &= \lim_{N \rightarrow \infty} \frac{1}{1 + \left(\frac{F(x_i)}{F(x_j)} \right)^N \gamma_{ij}}, \\ &= \begin{cases} \gamma_{ij}, & \text{if } F(x_i) < F(x_j), \\ 0, & \text{if } F(x_i) > F(x_j), \end{cases} \end{aligned}$$

and

$$\lim_{N \rightarrow \infty} P_{ii} = \lim_{N \rightarrow \infty} \left(\gamma_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)} \right)^N \gamma_{ij}} \right),$$

$$\begin{aligned}
&= \gamma_{ii} + \sum_{\substack{j=1 \\ j \neq i \\ F(x_i) > F(x_j)}}^n \gamma_{ij}, \\
&= 1 - \sum_{\substack{j=1 \\ j \neq i \\ F(x_i) < F(x_j)}}^n \gamma_{ij}.
\end{aligned}$$

Without loss of generality, assume that the cells of the SEGS are ordered according to decreasing fitness value, so that the index $I = 1$. The matrix $\lim_{N \rightarrow \infty} \mathbf{P}$ is therefore a lower triangular matrix.

We seek

$$\begin{aligned}
\lim_{N \rightarrow \infty} \boldsymbol{\sigma}_1 &= \lim_{N \rightarrow \infty} (\mathbf{I} - \mathbf{D}_1 \mathbf{P})^{-1} \mathbf{D}_1 \mathbf{1}, \\
&= (\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})^{-1} \mathbf{D}_1 \mathbf{1},
\end{aligned}$$

where $(\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})^{-1}$ always exists due to the following.

- (i) $\lim_{N \rightarrow \infty} \mathbf{P}$ is a lower triangular matrix with full rank. All of the lower triangular elements are non-zero.
- (ii) \mathbf{D}_1 is a lower triangular matrix with rank $n - 1$.
- (iii) $(\mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})$ is a matrix with zeros in row one, and elements that are equal to $\lim_{N \rightarrow \infty} \mathbf{P}$ in all other rows. Hence, $(\mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})$ has rank $n - 1$. Since this matrix is the product of lower triangular matrices, it is also lower triangular.
- (iv) $(\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})$ is a lower triangular matrix because it is the difference of lower triangular matrices. All lower triangular elements of this matrix are non-zero, with the matrix element $(\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})_{11} = 1$. Thus, $(\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})$ has full rank.

Since $(\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P})$ is a lower triangular matrix with full rank, the equation

$$(\mathbf{I} - \mathbf{D}_1 \lim_{N \rightarrow \infty} \mathbf{P}) \lim_{N \rightarrow \infty} \boldsymbol{\sigma}_1 = \mathbf{D}_1 \mathbf{1},$$

may be solved by the iterative process of forward substitution to obtain unique constant values of

$\lim_{N \rightarrow \infty} \sigma_{i1}$ for each i . For instance,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sigma_{11} &= 0, \\ \lim_{N \rightarrow \infty} \sigma_{21} &= \frac{1}{\gamma_{21}}, \\ \lim_{N \rightarrow \infty} \sigma_{31} &= \frac{1 + \gamma_{32}(\lim_{N \rightarrow \infty} \sigma_{21})}{\gamma_{31} + \gamma_{32}}, \\ &= \frac{1 + \frac{\gamma_{32}}{\gamma_{21}}}{\gamma_{31} + \gamma_{32}}, \\ \lim_{N \rightarrow \infty} \sigma_{41} &= \frac{1 + \gamma_{42}(\lim_{N \rightarrow \infty} \sigma_{21}) + \gamma_{43}(\lim_{N \rightarrow \infty} \sigma_{31})}{\gamma_{41} + \gamma_{42} + \gamma_{43}}, \\ &= \frac{1 + \frac{\gamma_{42}}{\gamma_{21}} + \gamma_{43} \left(\frac{1 + \frac{\gamma_{32}}{\gamma_{21}}}{\gamma_{31} + \gamma_{32}} \right)}{\gamma_{41} + \gamma_{42} + \gamma_{43}}, \end{aligned}$$

and so on. This completes the convergence part of the proof.

We next use induction on the cell index to show that σ_{i1} is a strictly decreasing function of N .

First, consider that

$$\sigma_{21} = \frac{\sum_{\substack{k=1 \\ k \neq 2}}^n P_{2k} (\sigma_{k1} + 1) + P_{22}}{1 - P_{22}}.$$

Hence,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sigma_{21} &= \lim_{N \rightarrow \infty} \frac{1 + \sum_{\substack{k=1 \\ k \neq 2}}^n P_{2k} \sigma_{k1}}{1 - P_{22}}, \\ &= \frac{\lim_{N \rightarrow \infty} \left(1 + \sum_{\substack{k=1 \\ k \neq 2}}^n P_{2k} \sigma_{k1} \right)}{\lim_{N \rightarrow \infty} (1 - P_{22})}, \\ &= \frac{1 + \lim_{N \rightarrow \infty} \left(\sum_{\substack{k=1 \\ k \neq 2}}^n P_{2k} \sigma_{k1} \right)}{1 - (1 - \gamma_{21})}, \end{aligned}$$

$$= \frac{1}{\gamma_{21}} + \frac{1}{\gamma_{21}} \lim_{N \rightarrow \infty} \left(\sum_{\substack{k=1 \\ k \neq 2}}^n P_{2k} \sigma_{k1} \right).$$

Comparing this expression to the result that was calculated by forward substitution above,

$\lim_{N \rightarrow \infty} \sum_{\substack{k=1 \\ k \neq 2}}^n P_{2k} \sigma_{k1}$ must decrease to 0 as N increases. Therefore, σ_{21} decreases as N increases.

For the induction hypothesis, assume that for any $s - 1$ where $2 \leq (s - 1) \leq (n - 1)$, we have that for all t where $2 \leq t \leq (s - 1)$, the mean hitting time σ_{t1} decreases with N . We now show that σ_{s1} is a decreasing function of N .

Consider that

$$\sigma_{s1} = \frac{\sum_{\substack{k=1 \\ k \neq s}}^n P_{sk} (\sigma_{k1} + 1) + P_{ss}}{1 - P_{ss}}.$$

Hence,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sigma_{s1} &= \lim_{N \rightarrow \infty} \frac{1 + \sum_{\substack{k=1 \\ k \neq s}}^n P_{sk} \sigma_{k1}}{1 - P_{ss}}, \\ &= \frac{\lim_{N \rightarrow \infty} \left(1 + \sum_{\substack{k=1 \\ k \neq s}}^n P_{sk} \sigma_{k1} \right)}{\lim_{N \rightarrow \infty} (1 - P_{ss})}, \\ &= \frac{1 + \lim_{N \rightarrow \infty} \left(\sum_{\substack{k=1 \\ k \neq s}}^n P_{sk} \sigma_{k1} \right)}{\sum_{k=1}^{s-1} \gamma_{sk}}, \\ &= \frac{1 + \lim_{N \rightarrow \infty} \left(\sum_{k=1}^{s-1} P_{sk} \sigma_{k1} + \sum_{k=s+1}^n P_{sk} \sigma_{k1} \right)}{\sum_{k=1}^{s-1} \gamma_{sk}}. \end{aligned}$$

Comparing this expression to the general result calculated by forward substitution,

$\lim_{N \rightarrow \infty} \sum_{k=s+1}^n P_{sk} \sigma_{k1}$ must decrease to 0 as N increases. By the induction hypothesis, $\lim_{N \rightarrow \infty} \sum_{k=1}^{s-1} P_{sk} \sigma_{k1}$ decreases with increasing N . Therefore, σ_{s1} is a decreasing function of N .

Hence, for all i where $2 \leq i \leq n$, an increase in the level of selectivity produces a corresponding decrease in the mean hitting time to the fittest cell, σ_{i1} , with $\lim_{N \rightarrow \infty} \sigma_{i1}$ approaching a unique constant value for each i . □

Acknowledgment

The authors would like to thank Hikaru Aono, Chang-Kwon Kang, Wei Shyy, and Patrick Trizila for their assistance with the surrogate and Theodorsen-Garrick models of flapping wing flight.

Bibliography

- [1] A. Borodin and R. El-Yaniv, *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.
- [2] A. Fiat and G. J. Woeginger, Eds., *Online Algorithms: The State of the Art*, ser. Lecture Notes in Computer Science. Springer, 1998, vol. 1442.
- [3] N. Ascheuer, M. Grötschel, S. O. Krumke, and J. Rambau, “Combinatorial online optimization,” in *Operations Research Proceedings 1998*, P. Kall and H.-J. Lüthi, Eds. Springer, 1999, pp. 21–37.
- [4] M. Grötschel, S. O. Krumke, and J. Rambau, Eds., *Online Optimization of Large Scale Systems*. Springer, 2001.
- [5] S. Albers, “Online algorithms: A survey,” *Mathematical Programming*, vol. 97, no. 1–2, pp. 3–26, July 2003.
- [6] K. B. Ariyur and M. Krstić, *Real-Time Optimization by Extremum-Seeking Control*. Wiley, 2003.
- [7] P. V. Hentenryck and R. Bent, *Online Stochastic Combinatorial Optimization*. MIT Press, October 2006.
- [8] J. E. Dennis, Jr. and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*. SIAM, 1996.

- [9] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. SIAM, 2000.
- [10] D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd ed. Kluwer Academic Publishers, 2003.
- [11] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.
- [12] H. W. Kuhn and A. W. Tucker, “Nonlinear programming,” in *Proceedings of the Second Berkeley Symposium of Mathematical Statistics and Probability*. University of California Press, 1951.
- [13] M. J. Atallah, Ed., *Algorithms and Theory of Computation Handbook*. CRC Press, 1999.
- [14] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 2nd ed. MIT Press, 2001.
- [15] S. M. Meerkov, “Mathematical theory of behavior — individual and collective behavior of retardable elements,” *Mathematical Biosciences*, vol. 43, no. 1–2, pp. 41–106, 1979.
- [16] P. Kabamba, “The von Neumann threshold of self-reproducing systems,” in *Proceedings of the 2008 ASME Dynamic Systems and Control Conference*, 20–22 October 2008.
- [17] S. Freeman, *Biological Science*, 2nd ed. Pearson Prentice Hall, 2005.
- [18] R. Diestel, *Graph Theory*, 3rd ed. Springer, 2005.
- [19] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*, 2nd ed. Springer, 2008.
- [20] P. Brémaud, *Markov Chains: Gibbs fields, Monte Carlo Simulation and Queues*. Springer, 1999.
- [21] D. Burd, “Plastic not fantastic,” in *Project Reports of the Canada Wide Science Fair*, 2008.

- [22] R. S. Sutton and A. G. Barto, *Reinforcement Learning: An Introduction*. MIT Press, 1998.
- [23] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, “Optimization by simulated annealing,” *Science*, vol. 220, no. 4598, pp. 671–680, 1983.
- [24] A. Corana, M. Marchesi, C. Martini, and S. Ridella, “Minimizing multimodal functions of continuous variables with the ‘simulated annealing’ algorithm,” *ACM Transactions on Mathematical Software*, vol. 13, no. 3, pp. 262–280, 1987.
- [25] D. E. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning*. Addison Wesley, 1989.
- [26] L. Davis, Ed., *Handbook of Genetic Algorithms*. Van Nostrand Reinhold, 1991.
- [27] M. Mitchell, *An Introduction to Genetic Algorithms*. MIT Press, 1996.
- [28] I. Rechenberg, “Evolutionsstrategie: Optimierung technischer systeme nach prinzipien der biologischen evolution,” Ph.D. dissertation, Technical University of Berlin, 1971.
- [29] H.-P. Schwefel, *Evolution and Optimum Seeking*. Wiley, 1995.
- [30] H.-G. Beyer and H.-P. Schwefel, “Evolution strategies: A comprehensive introduction,” *Natural Computing*, vol. 1, no. 1, pp. 3–52, 2002.
- [31] D. B. Fogel, *Evolutionary Computation: Toward a New Philosophy of Machine Intelligence*, 3rd ed. Wiley, 2006.
- [32] G. Rudolph, “Convergence analysis of canonical genetic algorithms,” *IEEE Transactions on Neural Networks*, vol. 5, no. 1, pp. 96–101, 1994.
- [33] D. B. Fogel, T. Bäck, and Z. Michalewicz, *Evolutionary Computation: Basic Algorithms and Operators*. CRC Press, 2000.
- [34] C. E. Shannon, “A mathematical theory of communication,” *Bell System Technical Journal*, vol. 27, pp. 379–423 and 623–656, July and October 1948.

- [35] D. F. Kerridge, "Inaccuracy and inference," *Journal of the Royal Statistical Society, Series B (Methodological)*, vol. 23, no. 1, pp. 184–194, 1961.
- [36] Y. A. Cengel and M. A. Boles, *Thermodynamics: An Engineering Approach*, 4th ed. McGraw-Hill, 2001.
- [37] R. K. Pathria, *Statistical Mechanics*, 2nd ed. Butterworth-Heinemann, 1996.
- [38] J. Bergelson and C. B. Purrington, "Surveying patterns in the cost of resistance in plants," *The American Naturalist*, vol. 148, no. 3, pp. 536–558, September 1996.
- [39] A. R. Kraaijeveld and H. C. J. Godfray, "Trade-off between parasitoid resistance and larval competitive ability in *Drosophila Melanogaster*," *Nature*, vol. 389, no. 6648, pp. 278–280, September 1997.
- [40] A. Bult and C. B. Lynch, "Breaking through artificial selection limits of an adaptive behavior in mice and the consequences for correlated responses," *Behavior Genetics*, vol. 30, no. 3, pp. 193–206, May 2000.
- [41] L. Xie and P. L. Klerks, "Responses to selection for cadmium resistance in the least killifish, *Heterandria Formosa*," *Environmental Toxicology and Chemistry*, vol. 22, no. 2, pp. 313–320, February 2003.
- [42] J. M. S. Burger, M. Kolss, J. Pont, and T. J. Kawecki, "Learning ability and longevity: A symmetrical evolutionary trade-off in *Drosophila*," *Evolution*, vol. 62, no. 6, pp. 1294–1304, June 2008.
- [43] K. J. Ryan and C. G. Ray, Eds., *Sherris Medical Microbiology: An Introduction to Infectious Diseases*, 4th ed. McGraw-Hill, 2004.
- [44] E. J. Gardner, M. J. Simmons, and D. P. Snustad, *Principles of Genetics*, 8th ed. Wiley, 1998.

- [45] P. Trizila, C.-K. Kang, M. Visbal, and W. Shyy, “A surrogate model approach in 2d versus 3d flapping wing aerodynamic analysis,” in *Proceedings of the 12th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference*, 10–12 September 2008.
- [46] P. Augustsson, K. Wolff, and P. Nordin, “Creation of a learning, flying robot by means of evolution,” in *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO 2002)*, 9–13 July 2002.
- [47] R. Hunt, G. S. Hornby, and J. D. Lohn, “Toward evolved flight,” in *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO 2005)*, 25–29 June 2005.
- [48] Y.-S. Shim and C.-H. Kim, “Evolving physically simulated flying creatures for efficient cruising,” *Artificial Life*, vol. 12, no. 4, pp. 561–591, 2006.
- [49] J. W. Roberts, J. Zhang, and R. Tedrake, “Motor learning at intermediate reynolds number: Experiments with policy gradient on a heaving plate,” 2009.
- [50] N. V. Queipo, R. T. Haftka, W. Shyy, T. Goel, R. Vaidyanathan, and P. K. Tucker, “Surrogate-based analysis and optimization,” *Progress in Aerospace Sciences*, vol. 41, pp. 1–28, 2005.
- [51] J. Yan, R. J. Wood, S. Avadhanula, M. Sitti, and R. Fearing, “Towards flapping wing control for a micromechanical flying insect,” in *Proceedings of the 2001 IEEE International Conference on Robotics and Automation*, 21–26 May 2001, pp. 3901–3908.
- [52] L. Schenato, “Analysis and control of flapping flight: from biological to robotic insects,” Ph.D. dissertation, University of California at Berkeley, 2003.
- [53] X. Deng, L. Schenato, and S. S. Sastry, “Flapping flight for biomimetic robotic insects: Part II -- flight control design,” *IEEE Transactions on Robotics*, vol. 22, no. 4, pp. 789–803, August 2006.
- [54] J.-B. Mouret, S. Doncieux, and J.-A. Meyer, “Incremental evolution of target-following neuro-controllers for flapping-wing animats,” in *From Animals to Animats 9*, ser. Lec-

- ture Notes in Computer Science, S. Nolfi, G. Baldassarre, R. Calabretta, J. C. T. Hallam, D. Marocco, J.-A. Meyer, O. Miglino, and D. Parisi, Eds. Springer, 2006, vol. 4095, pp. 606–618.
- [55] L. Weng, W. Cai, M. J. Zhang, X. H. Liao, and D. Y. Song, “Neural-memory based control of micro air vehicles (mavs) with flapping wings,” in *Advances in Neural Networks — ISNN 2007*, ser. Lecture Notes in Computer Science, D. Liu, S. Fei, Z.-G. Hou, H. Zhang, and C. Sun, Eds. Springer, 2007, vol. 4491, pp. 70–80.
- [56] Z. A. Khan and S. K. Agrawal, “Control of longitudinal flight dynamics of a flapping-wing micro air vehicle using time-averaged model and differential flatness based controller,” in *Proceedings of the 2007 American Control Conference*, 11–13 July 2007, pp. 5284–5289.
- [57] S.-J. Chung, M. Dorothy, and J. R. Stonerz, “Neurobiologically inspired control of engineered flapping flight,” in *Proceedings of the 2009 AIAA Infotech at Aerospace and 2009 AIAA Unmanned Unlimited Conference*, no. AIAA-2009-1929, 6–9 April 2009.
- [58] T. Theodorsen, “General theory of aerodynamic instability and the mechanism of flutter,” NACA, Tech. Rep. 496, 1935.
- [59] I. E. Garrick, “Propulsion of a flapping and oscillating airfoil,” NACA, Tech. Rep. 567, 1936.
- [60] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.
- [61] M. M. Gupta, L. Jin, and N. Homma, *Static and Dynamic Neural Networks: From Fundamentals to Advanced Theory*. Wiley, 2003.